APPROXIMATION BY QUOTIENTS OF RATIONAL INNER FUNCTIONS

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ABSTRACT. Let \( u \) be a continuous unimodular function on the \( n \)-dimensional torus \( T^n \). It is shown that \( u \) can be uniformly approximated by quotients of rational inner functions.

Let \( T \) denote the unit circle. Let \( u \) be a Lebesgue measurable function on \( T \) such that \( |u| = 1 \) a.e. In [1] Douglas and Rudin proved that: given \( \varepsilon > 0 \), there exist inner functions \( g_1 \) and \( g_2 \) such that \( ||u - g_1 g_2|| < \varepsilon \). (\( || \cdot || \) indicates the essential-sup norm.) In this paper we establish a continuous analogue of Douglas and Rudin's result on the torus \( T^n \). We show that, if \( v \) is a continuous unimodular function on \( T^n \), then \( v \) can be uniformly approximated by quotients of rational inner functions.

Let \( I \) and \( I^+ \) denote, respectively, the set of integers and the set of nonnegative integers. Let \( \alpha = (a, b, \ldots, x) \in I^n \). We will use \( z^\alpha \) to denote the function defined on \( T^n \) by \( z^\alpha(\xi_1, \xi_2, \ldots, \xi_n) = \xi_1^a \xi_2^b \cdots \xi_n^x \). A finite linear combination of the \( z^\alpha \), where the \( \alpha \)'s are taken from \( I^n \), will be called a polynomial. If \( p = \sum c(\alpha)z^\alpha \) is a polynomial, we will use the notation \( \alpha(p) \) to denote the \( n \)-tuple \( \langle a_1(p), \ldots, a_n(p) \rangle \), where \( a_i(p) \) denotes the maximum \( i \)th component of any \( \alpha \) satisfying \( c(\alpha) \neq 0 \), and we will use \( \tilde{p} \) to indicate the polynomial \( \sum c(\alpha)z^{(\alpha(p)-\alpha)} \). Clearly, polynomials are also well defined over the closure of the open unit polydisk \( D^n \). A rational inner function on \( T^n \) is a function of the form \( cz^n(\tilde{p}/p) \), where \( c \) is a constant with \( |c| = 1 \), where \( \alpha \in I^n \), and where \( p \) is a polynomial having no zeros on the closure of \( D^n \). (Our definition of rational inner function is essentially the same as the one in Rudin's book [3, p. 110].) Finally, let \( U \) and \( U_0 \) denote, respectively, the set of continuous unimodular functions on \( T^n \) and the set of continuous unimodular functions on \( T^n \) having continuous logarithms. Note that \( U \) and \( U_0 \) are both groups under the usual operation of (pointwise) multiplication of complex-valued functions.

**Proposition.** For each \( u \in U \), there is an \( \alpha \in I^n \) such that \( z^\alpha u \in U_0 \).

**Proof.** It can be shown that the factor group \( U/U_0 \) is isomorphic to the first Čech cohomology group \( H^1 \) of \( T^n \), where the coefficients are taken from \( I \). (This can be done quickly by applying the Arens-Royden theorem; see [2].)
It is an exercise in algebraic topology to show that $H^1$ is isomorphic to $I^n$. Since $\{z^nU_0|\alpha \in I^n\}$ is a free abelian group having $n$ generators, it follows that $U/U_0 = \{z^nU_0|\alpha \in I^n\}$.

**Theorem.** Let $V$ denote the closure in the topology of uniform convergence of functions in $U$ of the form $gg_1$, where $g$ and $g_1$ are rational inner functions. Then $V = U$.

**Proof.** Since $V$ is a subgroup of $U$ and since $z^n \in V$ for every $\alpha \in I^n$, it follows from the proposition above that $U_0 \subseteq V$ implies $V = U$. Suppose $u \in U_0$, then $u = e^{if}$, where $f$ is a real-valued continuous function on $T^n$. Hence, for each positive integer $m$, we have $u = (u_m)^n$, where $u_m = e^{if/m}$. By choosing $m$ sufficiently large, the real part of $u_m$ can be made uniformly close to 1. It follows that, in order to show that $U_0 \subseteq V$, it suffices to prove that every $u \in U_0$ of the form $u = (v)^2$, where $\Re v \geq \frac{1}{2}$, lies in $V$. Let $\varepsilon > 0$ be given. By the Stone-Weierstrass theorem there exists a polynomial $p$ and an $a \in I_n^m$ such that $\|v - (z^a)p\| < \varepsilon/3$, $\|(1/v) - 1/(z^a)p\| < \varepsilon/3$, $\|z^a p\|^{-1} < 2$, and $\frac{1}{4} < \Re z^a p$. It follows that

$$\|v^2 - (z^a p)/(z^a p)\| < \|v^2 - v/(z^a p)\| + \|v/(z^a p) - (z^a p)/(z^a p)\| < \varepsilon.$$ 

Note that $z^a p = \bar{z}^{a(p)}$ on $T^n$. Hence, $(z^{a(p)}/(z^a p)) = z^{2a - a(p)} \bar{p}/p$. Thus, the proof will be completed if we can show that $p$ has no zeros in the closure of $D^n$. Note that the function $\Re z^a p$ is well defined on the closure of $D^n$ and is harmonic in each variable on $D^n$. It follows from the minimum principle for harmonic functions that $\Re z^a p > \frac{1}{4}$ on the closure of $D^n$. In particular $p$ cannot have a zero on the closure of $D^n$.

**References**


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