MULTIPLIERS OF $A^*$-ALGEBRAS

DAVID L. JOHNSON AND CHARLES D. LAHR

Abstract. Let $A$ be an $A^*$-algebra of the first kind with $C^*$-algebra completion $\mathfrak{A}$. It is known that if $A$ is dual then $A^2$ is dense in $A$ and the Banach algebras $M_L(A)$ and $M_L(\mathfrak{A})$ of left multipliers of $A$ and $\mathfrak{A}$ are algebra isomorphic. In this note it is proved that $M_L(A)$ and $M_L(\mathfrak{A})$ are topologically algebra isomorphic when $A$ is an arbitrary $A^*$-algebra of the first kind such that $A^2$ is dense in $A$. As a consequence, it follows that every left multiplier of a replete Hilbert algebra $A$ is automatically continuous.

1. Introduction. Let $(A, \| \cdot \|, | \cdot |)$ be an $A^*$-algebra where $\| \cdot \|$ is a Banach algebra norm, and $| \cdot |$ is the auxiliary norm [10, p. 181]. The Banach algebra $A = (A, \| \cdot \|)$ is of the first kind if $A$ is a $*$-ideal of its $C^*$-algebra completion $\mathfrak{A} = (\mathfrak{A}, | \cdot |)$. In this case, since $\| \cdot \|$ majorizes $| \cdot |$ [10, Corollary 4.1.16], $A$ is a Banach $\mathfrak{A}$-bimodule (i.e., there exists $K > 0$ such that $\|wa\| \leq K|w|\|a\|$, $\|aw\| \leq K\|a\|\|w\|$, for all $a \in A$, $w \in \mathfrak{A}$ [3, Proposition 2.2, Theorem 2.3]). A left multiplier of an algebra $B$ is a linear map $T: B \to B$ such that $T(xy) = (Tx)y$, for all $x, y \in B$; every left multiplier of a semi-simple Banach algebra is continuous [6]. For $A$ (resp., $\mathfrak{A}$) as above, let $M_L(A)$ (resp., $M_L(\mathfrak{A})$) be the Banach algebra of all (automatically continuous, since $A$, $\mathfrak{A}$ are semisimple [10, Theorem 4.1.19]) left multipliers of $A$ (resp., $\mathfrak{A}$).

Now, if $A$ is a dual $A^*$-algebra of the first kind, then $A^2 = \text{sp}\{ab: a, b \in A\}$ is dense in $A$ [10, Corollary 2.8.3], and $M_L(A)$, $M_L(\mathfrak{A})$ are algebra isomorphic [12, Theorem 5.1], [11, Theorem 4.2]. In this paper, we show that if $A$ is an arbitrary $A^*$-algebra of the first kind with $A^2$ dense in $A$, then $M_L(A)$ is topologically algebra isomorphic to $M_L(\mathfrak{A})$. As an application of this result, it is proved that every left multiplier of a replete Hilbert algebra $A = (A, \| \cdot \|)$ is automatically continuous. This result appears to be new even for full Hilbert algebras. In general, the Hilbert space norm $\| \cdot \|$ is not an algebra norm on $A$, nor is $A$ complete in this norm. The authors know of no other example of an automatic continuity result for multipliers of a non-Banach nonnormed algebra.

2. Main result. As in [3], we denote the spectrum of an element $x$ in a Banach algebra $B$ by $\text{Sp}_B(x)$ and its spectral radius by $r_B(x)$.

Theorem 1. Let $A$ be an $A^*$-algebra of the first kind with $A^2$ dense in $A$, and...
let \( \mathcal{A} \) be the C*-algebra completion of \( A \). Then \( M_L(A) \) is topologically algebra isomorphic to \( M_L(\mathcal{A}) \).

**Proof.** First, since \( A^2 \) is dense in \( A \), the Hewitt-Cohen factorization theorem \([5, \text{Theorem 32.22}]\) implies that \( A = \mathcal{A} \cdot A = \{ wa : w \in \mathcal{A}, a \in A \} \).

Thus, if \( T \in M_L(\mathcal{A}) \), then \( T' = T|_A \) maps \( A \) into \( A \) (since \( T'(A) = T(A) = T(\mathcal{A} \cdot A) = T(\mathcal{A}) \cdot A \subseteq \mathcal{A} \cdot A = A \)); hence, \( T' \in M_L(A) \). Next, because \( A \) is a dense \(*\)-ideal of \( \mathcal{A} \) and \( \mathcal{A} \cdot A = A \), \( A \) possesses a (left) approximate identity \( \{ u_a \}_a \) such that \( |u_a| \leq 1 \), for all \( a [4, \text{Proposition 1.7.2}] \). Therefore, for each \( a \) in \( A \),

\[
\| T'a \| = \lim_a \| T'(u_a a) \| = \lim_a \| (Tu_a)a \|
\leq \sup_a K|Tu_a| \|a\| < K\|T\| \|a\|;
\]

whence \( \| T' \| < K\|T\| \). Now, \( A \) is dense in \( \mathcal{A} \), so the continuous linear map \( T \mapsto T' \) from \( M_L(\mathcal{A}) \) into \( M_L(A) \) is a vector space isomorphism, and is easily seen to be an algebra isomorphism as well. Thus, it remains only to show that the map \( T \mapsto T' \) is surjective.

Since \( A \) is a left ideal of \( \mathcal{A} \) and of \( M_L(A) \), \( \text{Sp}_\mathcal{A}(a) \cup \{ 0 \} = \text{Sp}_{M_L(A)}(a) \cup \{ 0 \} \) [3, proof of Proposition 4.1]; hence \( \nu_\mathcal{A}(a) = \nu_{M_L(A)}(a) \), for each \( a \) in \( A \). Consequently, if \( L_a \) denotes left multiplication by \( a \) in \( M_L(A) \), then

\[
|a|^2 = |a^*a| = \nu_\mathcal{A}(a^*a) = \nu_{M_L(A)}(a^*a) < \| L_a^*a \| = \sup \{ \| a^*ab \| : b \in A, \| b \| < 1 \}
\leq K|a^*| \sup \{ \| ab \| : b \in A, \| b \| < 1 \} = K|a| \| L_a \|,
\]

and so \( |a| < K\|L_a\| < K^2|a| \), for all \( a \) in \( A \). Now, let \( S \in M_L(A) \) be given. Then, for each \( a \) in \( A \),

\[
|Sa| < K\|L_{Sa}\| = K\|SL_a\| < K\|S\| \| L_a \| < K^2\|S\| \|a\|;
\]

therefore, \( S \) extends uniquely to a continuous linear operator \( T \) on \( \mathcal{A} \) with \( \| T \| < K^2\|S\| \). It follows immediately that \( T \in M_L(\mathcal{A}) \) and that \( T' = S \). \( \square \)

Observe that if the \( A^* \)-algebra \( A \) is an isometric Banach \( \mathcal{A} \)-bimodule (i.e., if the constant \( K = 1 \)), then \( M_L(A) \) is isometrically algebra isomorphic to \( M_L(\mathcal{A}) \). In addition, under the assumptions of Theorem 1, it follows mutatis mutandis that the Banach algebras \( M_R(A) \), \( M_R(\mathcal{A}) \) of all right multipliers of \( A \), \( \mathcal{A} \) are topologically algebra isomorphic. This fact, together with Theorem 1, implies that the Banach algebras \( M(A) \), \( M(\mathcal{A}) \) of all double multipliers of \( A \), \( \mathcal{A} \) are also topologically algebra isomorphic.

3. Application. If \( A = (A, \| \cdot \|) \) is a replete Hilbert algebra (in particular, every full Hilbert algebra is replete; see [7], [13] for definitions), then in the so-called Rieffel norm \( \| \cdot \|_r, A_r = (A, \| \cdot \|_r) \) is an \( A^* \)-algebra of the first kind such that \( A_r^2 \) is dense in \( A \) [7, Theorem 4.1]. Thus, Theorem 1 applies, and yields the following result. Let \( M_L(A) \) be the Banach algebra of all continuous left multipliers of \( A = (A, \| \cdot \|) \).
Theorem 2. If \( A \) is a replete Hilbert algebra with C*-algebra completion \( \mathcal{A} = (\mathcal{A}, | \cdot |) \), and \( A_r = (A, \| \cdot \|_r) \), then \( M_L(A) = M_L(A_r) \) as Banach algebras. Hence, every left multiplier of \( A \) is automatically continuous.

Proof. First, since \( A = A_r \) as abstract algebras, and \( A_r \) is a semisimple Banach algebra, \( M_L(A) \subseteq M_L(A_r) \) set-theoretically. On the other hand, \( A_r \) is an isometric Banach \( \mathcal{A} \)-bimodule; hence, \( M_L(A_r) \) is isometrically algebra isomorphic to \( M_L(\mathcal{A}) \) by Theorem 1. Further, since \( \mathcal{A} \subseteq \mathcal{B}(H) \), the bounded linear operators on the Hilbert space completion \( H = (H, \| \cdot \|) \) of \( A = (A, \| \cdot \|_r) \), it follows from [1, Proposition 4.2] that \( M_L(\mathcal{A}) \) is isometrically algebra isomorphic to a closed subalgebra of \( \mathcal{B}(H) \). Hence, every left multiplier \( T \) in \( M_L(A_r) \) is continuous on \( A = (A, \| \cdot \|) \) (i.e., \( M_L(A_r) \subseteq M_L(A) \)) and, in addition, the Banach algebras \( M_L(A) \) and \( M_L(A_r) \) have the same norm. Finally, if \( T \) is a left multiplier of \( A = A_r \), then (since \( A_r \) is semisimple) \( T \in M_L(A_r) = M_L(A) \). \( \square \)

In an earlier manuscript [8], the authors gave a proof of Theorem 2 in the spirit of [9], [2]. We would like to thank G. F. Bachelis for his helpful comments regarding [8], consideration of which eventually led to Theorem 1 in its present generality.

References

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Department of Mathematics, University of Southern California, Los Angeles, California 90007

Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755