HUREWICZ ISOMORPHISM THEOREM FOR STEENROD HOMOLOGY

Y. KODAMA AND A. KOYAMA

Abstract. For a pointed compactum \((X, x)\), a natural homomorphism \(\xi_n\) from the Quigley’s approaching group \(\pi_n(X, x)\) to the Steenrod homology group \(\tilde{H}_{n+1}(X)\) is defined. A shape theoretical condition under which \(\xi_n\) is an isomorphism is obtained. For every pointed \(S^n\)-like continuum \((X, x)\), \(\xi_n\) is an isomorphism for \(n \neq 2\) and \(\xi_2\) is an isomorphism if and only if \(X\) is movable.

1. Introduction. Let \(X\) be a compact metric space and let \(x \in X\). Denote by \(\tilde{H}_n(X)\) the \(n\)-dimensional Steenrod homology group of \(X\) defined by Steenrod [14] and by \(\pi_n(X, x)\) the \(n\)-dimensional approaching group of \((X, x)\) defined by Quigley [13]. For each \(n\), there exists a natural homomorphism \(\xi_n: \pi_n(X, x) \rightarrow \tilde{H}_{n+1}(X)\). The purpose of this note is to establish a shape theoretical condition for \(X\) under which \(\xi_n\) is an isomorphism.

Throughout the paper all spaces are metrizable and maps are continuous. We denote by \(J\) the directed set of nonnegative integers.

2. \((m, n)\)-movability. Let \((K, v)\) be a pointed polyhedron and let \((Y, y)\) be a pointed space. For \(k \in J\), a map \(f: (Y, y) \rightarrow (K, v)\) is said to be \(k\)-deformable if there exists a map \(g: (Y, y) \rightarrow (K, v)\) such that \(f \simeq g\) rel \(\{y\}\) and \(g(Y)\) is contained in a combinatorial \(k\)-skeleton of a triangulation of \(K\). For a pair \((n, r)\), \(n, r \in J\), a pointed compactum \((X, x)\) is said to be \((m, n)\)-movable if there exists an inverse sequence \(\{(A_i, x_i), \pi_i\}\) consisting of pointed finite polyhedra \((A_i, x_i), i \in J\), and bonding maps \(\pi: (X_j, x_j) \rightarrow (X_i, x_i), i < j, i, j \in J\), satisfying the following conditions:

\[
(2.1) \quad (X, x) = \lim_{ \leftarrow } (X_i, x_i).
\]

\[
(2.2) \quad For every i \in J, there exists j \in J, j \geq i, such that for every k \in J, k \geq i, and every n-deformable map f: (S^n, s) \rightarrow (X_j, x_j), where (S^n, s) is a pointed m-sphere, there exists an n-deformable map g: (S^n, s) \rightarrow (X_k, x_k) such that \(\pi_k f \simeq g\) rel \(s\).
\]

For a pointed compactum \((X, x)\), denote by \(\text{Sh}(X, x)\) the shape of \((X, x)\) defined by Borsuk [2]. The \((m, n)\)-movability of \((X, x)\) is a hereditary shape invariant in the sense of Borsuk, that is, we have
Theorem 1. Let \((X, x)\) and \((Y, y)\) be pointed compacta. If \(\text{Sh}(X, x) > \text{Sh}(Y, y)\) and \((X, x)\) is \((m, n)\)-movable, then \((Y, y)\) is \((m, n)\)-movable.

The proof is given by a standard technique (cf. [7, Theorem 1]) and we omit it.

For \(n \in J\), the pointed \(n\)-movability of \((X, x)\) is defined in a similar way to one given by Borsuk [1], that only the category of pointed compacta is considered. The following lemma is obvious from [7].

Lemma 1. A pointed \(n\)-movable compactum is \((m, n)\)-movable for every \(m \in J\).

Since the pointed movability (cf. [2, p. 166]) implies the pointed \(n\)-movability for every \(n \in J\), we have

Corollary 1. A pointed movable compactum is \((m, n)\)-movable for \(m, n \in J\).

Example 1. Let \(n \in J\), \(n > 0\). Let \(\{X_i, p_{i+1}\}\) be an inverse sequence consisting of \(n\)-spheres \(X_i\), \(i \in J\), such that each bonding map \(p_{i+1}: X_{i+1} \to X_i\) is of degree 2. Let \(X(n)\) be the limit space of \(\{X_i\}\) and \(x \in X(n)\). Then we have

(2.3) if \(n \neq 2\), \((X(n), x)\) is \((n + 1, n)\)-movable,
(2.4) \((X(2), x)\) is not \((3, 2)\)-movable.

To see (2.3) it is enough to note that \(\pi_2(S^1) = 0\) and \(\pi_{n+1}(S^n) = \mathbb{Z}_2\) for \(n > 3\), where \(\pi_i(Y)\) is the \(i\)-homotopy group of \(Y\). (Note that \(X(1)\) is \((k, 1)\)-movable for each \(k > 1\).)

Assertion (2.4) follows from \(\pi_3(S^2) = \mathbb{Z}\) (see [6, Chapter VI, Lemma 1.2]) and the definition of the \((3, 2)\)-movability.

Since \((X(n), x)\) is not pointed \(n\)-movable, the converse assertion of Lemma 1 or Corollary 1 does not generally hold.

3. Hurewicz isomorphism theorem. For a compactum \(X\), let \(H_n(X)\) be the homology group of the regular \(n\)-cycles of \(X\) defined by N. E. Steenrod [14]. A beautiful description of \(H_n(X)\) was given by J. Milnor [11]. For a pointed compactum \((X, x)\), Quigley [13] defined the approaching group \(\pi_n(X, x)\). To define a natural homomorphism \(\xi_n: \pi_n(X, x) \to H_{n+1}(X)\), consider \(X\) as a subset of the Hilbert cube \(Q\). For an element \(\alpha \in \pi_n(X, x)\), let \(f: R^+ \times (S^n, s) \to (Q, x)\) be an approaching \(n\)-mapping representing in the sense of Quigley [13], where \(R^+ = \{t: 0 < t < \infty\}\). Let \(D^{n+1}\) be an \((n + 1)\)-ball whose boundary is \(S^n\) and put \(K = \{0\} \times D^{n+1} \cup R^+ \times S^n\). Let \(g: K \to Q\) be an extension of \(f\). Since \(K\) is an infinite \((n + 1)\)-cycle, the triple \((K, f, K)\) is an infinite \((n + 1)\)-cycle regular to \(X\) in the sense of Steenrod [14, p. 837]. Let \(\beta\) be the element of \(H_{n+1}(X)\) represented by \((K, f, K)\). Obviously \(\beta\) is uniquely determined by the element \(\alpha\). Define \(\xi_n: \pi_n(X, x) \to H_{n+1}(X)\) by \(\xi_n(\alpha) = \beta\). It is easy to show that \(\xi_n\) is a natural homomorphism.
Theorem 2. Let \((X, x)\) be a pointed compactum and let \(n \in J, n > 1\). If
\((X, x)\) is \((n + 1, n)\)-movable and approximately \(k\)-connected for \(k = 0, 1, \ldots, n - 1\) [2, p. 145], then \(\xi_n : \pi_n(X, x) \to H_{n+1}(X)\) is an isomorphism.

Proof. Let \(\{(X_i, x_i), p_{i+1}\}\) be an inverse sequence consisting of pointed polyhedra such that \((X, x) = \lim \{(X_i, x_i), p_{i+1}\}\). Let \(\{H_k(X_i)\}\) be the inverse sequence consisting of the \(k\)-homology groups. Similarly let \(\{\pi_k(X_i, x_i)\}\) be the inverse sequence consisting of \(k\)-homotopy groups of \((X_i, x_i), i \in J\). From the proofs of [14, Theorem 7], [11, Theorem 4] and [15, Theorem 2] it is seen that the homomorphism \(\xi_n\) induces homomorphisms \(\mu\) and \(\eta_n\) such that the following diagram is commutative,

\[
0 \to \lim \{(\pi_{n+1}(X_i, x_i)) \to \pi_n(X, x) \to \pi_n(X, x) \to 0
\]

\[
\downarrow \text{\(\mu\)} \quad \downarrow \text{\(\xi_n\)} \quad \downarrow \text{\(\eta_n\)}
\]

\[
0 \to \lim \{H_{n+1}(X_i)\} \to H_{n+1}(X) \to \hat{H}_n(X) \to 0
\] (3.1)

Here \(\hat{H}_n(X)\) is the Čech \(n\)-homology group of \(X, \pi_n(X, x)\) is the \(n\)th fundamental group defined by Borsuk [2, Chapter XII] and \(\lim(1)\) is the first derived functor of the inverse limit functor \(\lim\). The homomorphism \(\mu\) is induced by the Hurewicz homomorphism \(\mu_j : \pi_{n+1}(X_i, x_i) \to H_{n+1}(X_i), j \in J\), and \(\eta_n\) is the limit Hurewicz homomorphism in the sense of Kuperberg [9, p. 26]. The exactness of the top row of diagram (3.1) follows from Theorem 2 of Watanabe [15] (cf. Grossman [5] or Edwards and Hastings [4, 5.2.1]). Milnor [11, Theorem 4] proved the bottom row of (3.1) is exact. Since \((X, x)\) is approximatively \(k\)-connected for \(k = 0, 1, \ldots, n - 1\), by [9, Theorem 3.2] \(\eta_n\) is an isomorphism. It remains to prove that \(\mu\) is an isomorphism. To show it, note that we may assume every \((X_i, x_i), i \in J\), is \((n - 1)\)-connected. (This is proved by the same way as in Lemma (1.6) of Nowak [12]). Then the Hurewicz homomorphism \(\mu_j : \pi_{n+1}(X_i, x_i) \to H_{n+1}(X_i), j \in J\) is onto by [6, Theorem 2.6]. Let \(G_i = \text{Kernel} \mu_i, i \in J\). Then \(\{G_i\}\) forms an inverse sequence. Consider the following exact sequence in the category pro-\(G\) of pro-groups:

\[
0 \to \{G_i\} \xrightarrow{(j_i)} \{\pi_{n+1}(X_i, x_i)\} \xrightarrow{} \{H_{n+1}(X_i)\} \xrightarrow{} 0,
\] (3.2)

where \(j_i : G_i \to \pi_{n+1}(X_i, x_i)\) is the inclusion homomorphism, \(i \in J\), and 0 means a zero object in pro-\(G\). By [3, p. 256] the sequence (3.2) induces the exact sequence:

\[
\lim(1)\{G_i\} \xrightarrow{(\xi_i)} \lim(1)\{\pi_{n+1}(X_i, x_i)\} \xrightarrow{} \lim(1)\{H_{n+1}(X_i)\} \xrightarrow{} 0.
\] (3.3)

Let \(f : (S^{n+1}, s) \to (X_i, x_i)\) be a map representing an element \(\alpha \in \pi_{n+1}(X_i, x_i)\). Then \(\alpha \in G_i\) if and only if \(f\) is \(n\)-deformable. From this fact and the \((n + 1, n)\)-movability of \((X, x)\) it follows that the inverse sequence \(\{G_i\}\) satisfies the Mittag-Leffler condition. Thus \(\lim(1)\{G_i\} = 0\) by [3, p. 256]. The
exactness of the sequence (3.3) shows that \( \mu \) is an isomorphism. This completes the proof.

**Corollary 2.** Let \((X, x)\) be a pointed movable compactum such that \( \pi_i(X, x) = 0 \) for \( i = 0, 1, \ldots, n - 1, n > 1 \). Then \( \pi_n(X, x) = H_n(X) = H_n(X) \).

This is obvious from Corollary 1, Theorem 2 and Borsuk [2, Chapter V, Theorem (10.1)].

**Corollary 3.** Let \((X, x)\) be approximatively connected for \( k = 0, 1, \ldots, n - 1 \). If there exists an inverse sequence \( \{ (X_i, x_i) \} \) such that \( (X, x) = \lim (X_i, x_i) \) and \( \lim (\pi_n(X_i, x_i)) = 0 \), then \( \xi_n: \pi_n(X, x) \rightarrow H_{n+1}(X) \) is an isomorphism.

**Proof.** By [8], \((X, x)\) is pointed \( S^n \)-movable. Since the approximative \( k \)-connectedness for \( k = 0, 1, \ldots, n - 1 \) and the pointed \( S^n \)-movability imply the pointed \( n \)-movability, the corollary follows from Lemma 1 and Theorem 2.

**Corollary 4.** Let \( X \) be an \( S^n \)-like continuum and let \( n \neq 2 \). Then, for every point \( x \in X \), the homomorphism \( \xi_n: \pi_n(X, x) \rightarrow H_{n+1}(X) \) is an isomorphism.

**Proof.** By (2.3) of Example 1, \((X, x)\) is \((n + 1, n)\)-movable. Thus, if \( n > 2 \) the corollary follows from Theorem 2. For \( n = 1 \), consider the diagram (3.1). Obviously \( \lim (\pi_2(X_i, x_i)) = \lim (H_2(X_i)) = 0 \) and \( \eta_1 \) is an isomorphism. Thus \( \xi_1 \) is an isomorphism.

**Corollary 5.** Let \( X \) be an \( S^2 \)-like continuum and let \( x \in X \). Then the following conditions are equivalent:

- (3.4) \((X, x)\) is pointed movable.
- (3.5) \( \xi_2: \pi_2(X, x) \rightarrow H_3(X) \) is an isomorphism.

**Proof.** The implication (3.4) \( \rightarrow \) (3.5) is a consequence of Theorem 2. To prove (3.5) \( \rightarrow \) (3.4), assume that \((X, x)\) is not pointed movable. Let \( \{ (X_i, x_i), p_{i,i+1} \} \) be an inverse sequence consisting of 2-spheres such that \( (X, x) = \lim (X_i, x_i) \). By Mardešić and Segal [10, Theorem 4], \( X \) is neither of trivial shape nor of the shape of \( S^2 \). Then, for an infinite number of \( i \in J \), the bonding map \( p_{i,i+1} \) has degree \( > 2 \). Since \( \pi_3(X_i, x_i) = Z \), we have \( \lim (\pi_3(X_i, x_i)) \neq 0 \). On the other hand, \( \lim (H_3(X_i)) = 0 \). Thus the homomorphism \( \mu \) in (3.1) is not an isomorphism. Hence \( \xi_2 \) is not an isomorphism.

Example 1 and Corollary 5 imply that we cannot omit the \((n + 1, n)\)-movability or the approximative connectivity of \((X, x)\) in Theorem 2. Finally, we give an example of a pointed continuum \((X, x)\) such that \((X, x)\) is approximatively 1-connected but not pointed \((3, 2)\)-movable, and \( \xi_2 \) is an isomorphism.

**Example 2.** For \( n \in J \), let \( X_n \) be a one point union of a 2-sphere \( S_n \) and
3-spheres $S_{n,i}$, $i = 1, 2, \ldots, n$, with the base point $x_n$. Let $p_{n,n+1} : (X_{n+1}, x_{n+1}) \rightarrow (X_n, x_n)$ be a map such that

$p_{n,n+1}|S_{n+1}$ is a map from $S_{n+1}$ to $S_n$ with degree 2,

$p_{n,n+1}|S_{n+1,1}$ is the Hopf map from $S_{n+1,1}$ to $S_n$,

$p_{n,n+1}|S_{n+1,i}, i = 2, \ldots, n + 1$, is a homeomorphism from $S_{n+1,i}$ to $S_{n,i-1}$.

Let $(X, x)$ be the limit of the inverse sequence $\{(X_n, x_n), p_{n,n+1}\}$. As shown in Example 1, $(X, x)$ is not $(3, 2)$-movable. Obviously $\pi_3(X, x) = H_3(X) = 0$. Since both of the inverse sequences $\{\pi_3(X_n, x_n)\}$ and $\{H_3(X_n)\}$ satisfy the Mittag-Leffler condition, by diagram (3.1) we have $\pi_2(X, x) = H_2(X) = 0$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TSUKUBA, IBARAKI, JAPAN