THE JACOBSON RADICAL OF THE GROUP ALGEBRA OF A FINE GROUP

SURINDER SINGH BEDI

Abstract. Let $K$ be a field of characteristic $p 
eq 0$ and $G$ a finite group such that $p 
mid o(G)$. Suppose $G$ is a Frobenius group with a Sylow $p$-subgroup $P$ as a complement. Then we have proved that

$$JK(G) = \bigcap_{x \in G} JK(P^x)K(G).$$

We have given an example to show that equality does not hold in general.

0. Introduction. Let $G$ be a finite group and $K$ a field of characteristic $p \neq 0$. If $p \nmid o(G)$ then $JK(G) = 0$ and if $p | o(G)$ then $JK(G) \neq 0$ [2, Theorem 1.4.1]. If $G$ is a $p$-group then [2, Theorem 2.3.2] $JK(G) = w(K(G))$, the augmentation ideal of $K(G)$. More generally [5, Theorem 16.6], if $G$ is a group having unique Sylow $p$-subgroup $P$, then

$$JK(G) = w(K(P))K(G).$$

So one asks the following question:

Is $JK(G) = \bigcap P JK(P)K(G)$ where $P$ ranges over Sylow $p$-subgroups of $G$?

We prove (Theorem 1 and Corollary 1) this to be the case for a group $G$ having a normal subgroup $G_0$ such that $p \nmid (G : G_0)$ and that $G_0$ is a Frobenius group having a Sylow $p$-subgroup $P$ as its complement subgroup. An example due to Passman shows that this equality is false in general.

1. Definitions and preliminaries. A finite group $G$ is said to be a Frobenius group with complement $H$ if $H$ is a subgroup of $G$ such that (1) $\{e\} \subseteq H \subseteq G$ and (2) $xHx^{-1} \cap H = \{e\}$ for every $x \in G - H$. A finite group $G$ is a Frobenius group iff $G$ is isomorphic to a transitive permutation group such that subgroup fixing any of the letters is nontrivial and each permutation $\neq e$ fixes at most one letter [1, Chapter 2, p. 37], [4, p. 57]. The most important fact regarding these groups is the following:

$$N = \{x \in G|C(x) \cap H = \emptyset\} \cup \{e\}$$

$$= \left(G - \bigcap_{x \in G} xHx^{-1}\right) \cup \{e\}$$

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is a normal subgroup of order \((G : H)\) and \(G = NH\) [1, Theorem 7.5]. The subgroup \(N\) is called the Frobenius kernel of \(G\). If \(H = P\), Sylow \(p\)-subgroup of \(G\) then \(p \mid o(N)\) because \(o(N) = (G : P)\). The number of distinct Sylow \(p\)-subgroups of \(G\) equals \(o(N)\).

We say \(x \in G\) is a \(p\)-element provided \(o(x) = p^i\) for some \(i > 0\). The \(p\)-trace of \(\sum a_xx \in K(G)\) is defined to be

\[
\text{tr}_p(\sum a_xx) = \sum_{x \text{ is a } p\text{-element}} a_x.
\]

**Lemma 1.** If \(\alpha \in K(G)\) is nilpotent then \(\text{tr}_p(\alpha) = 0\).

For proof see [6, Lemma 2.3.3].

**Lemma 2.** Suppose \(N\) and \(H\) are subgroups of the finite group \(G\) with \(N \triangleleft G\) and \(G = NH\). If \(\hat{N} = \sum_{n \in N} n\), then

\[
JK(H) \hat{N} \subset \left( \bigcap_{x \in G} JK(H^x)K(G) \right) \cap JK(G).
\]

**Proof.** Observe that \(\hat{N}\) is central in \(K(G)\) and \(n\hat{N} = \hat{N}\) for all \(n \in N\). Let \(x \in G = NH\) so \(x = nh\), where \(n \in N\) and \(h \in H\). Then

\[
JK(H^x) \hat{N} = h^{-1}n^{-1}JK(H)nh\hat{N} = h^{-1}JK(H)h\hat{N} = JK(H)\hat{N}.
\]

Hence \(JK(H)\hat{N} \subset \bigcap_{x \in G} JK(H^x)K(G)\). Since \((JK(H)\hat{N})x = x(JK(H)\hat{N})\) for all \(x \in G\), \(JK(H)\hat{N}\) generates a nilpotent two sided ideal of \(K(G)\). Thus \(JK(H)\hat{N} \subset JK(G)\).

2. **Main result.** From now on, we assume that \(K\) is an arbitrary field of characteristic \(p \neq 0\), \(G\) a finite group such that \(p \mid o(G)\), and \(P\) a fixed Sylow \(p\)-subgroup of \(G\).

**Theorem 1.** Suppose \(G\) is a Frobenius group with Sylow \(p\)-subgroup \(P\) as a complement. Then \(JK(G) = \bigcap_{x \in G} JK(P^x)K(G)\).

**Proof.** We shall prove this in two parts by showing that each side equals \(JK(P)\hat{N}\) where \(N\) is the Frobenius kernel and \(\hat{N}\) denotes the element \(\sum_{x \in N} x\).

First we shall show that \(JK(G) = JK(P)\hat{N}\). (Wallace [7] proved, using modular representation theory, that if \(G\) is a group such that \(G'P\) is a Frobenius group with \(G'\) as the Frobenius kernel and \(P\) as a complement then \(JK(G'P) = JK(P)\hat{N}'\). Our result is stronger and its proof uses simpler techniques.) By Lemma 2, \(JK(P)\hat{N} \subset JK(G)\). Conversely, let \(\alpha \in JK(G)\). Let

\[
\alpha = \sum_{x \in N} a_xx + \sum_{y \in G - N} a_yy.
\]

Since \(JK(G)\) is nilpotent \(\alpha\) is nilpotent. Thus by Lemma 1, we have \(\text{tr}_p(\alpha) = 0\). Since \(p \mid o(N)\) no element \(x \neq 1\) in \(N\) is a \(p\)-element. If \(y \in G - N\) then \(y\)
belongs to some Sylow $p$-subgroup of $G$. Thus $y$ is a $p$-element. Therefore $\text{tr}_p(\alpha) = 0$ gives

$$a_1 + \sum_{y \in G - N} a_y = 0.$$ 

Since $JK(G) \subset w(K(G))$ we have

$$\sum_{x \in N} a_x + \sum_{y \in G - N} a_y = 0$$

and hence $a_1 = \sum_{x \in N} a_x$. Now let $n \in N$. Then $an^{-1} \in JK(G)$ and

$$an^{-1} = \sum_{x \in N} a_x x n^{-1} + \sum_{y \in G - N} a_y y n^{-1}.$$ 

Therefore, by the same arguments, $a_n = \sum_{x \in N} a_x$ and hence $a_1 = a_n$ for all $n \in N$. Let $h \in P$. Then $ah^{-1} \in JK(G)$ and $ah^{-1} = (\sum_{x \in N} a_x x) h^{-1} + \alpha'$ where $\text{support}(\alpha') \subset G - N$. Then by the above argument, the coefficients of the elements of $N$ in $ah^{-1}$ are all equal. Therefore $a_n = a_nh$ for all $n \in N$.

Hence $\alpha = \sum_{h \in P} a_h \hat{N}h$. Since $\alpha \in JK(G) \subset w(K(G))$ we have $(\sum_{h \in P} a_h) o(N) = 0$ and so $\sum_{h \in P} a_h = 0$ because $o(N) \neq 0$ in $K$. Thus $\sum_{h \in P} a_h h \in w(K(P)) = JK(P)$. Hence $\alpha \in JK(P)\hat{N}$.

Now we shall prove that $\sum_{x \in G} JK(P^x)K(G) = JK(P)\hat{N}$. By Lemma 2, we have

$$JK(P)\hat{N} \subset \bigcap_{x \in G} JK(P^x)K(G).$$

For the reverse inclusion, let $I = \bigcap_{x \in G} JK(P^x)K(G)$. Let $P_1 = P, P_2, \ldots, P_i$ be all the distinct Sylow $p$-subgroups of $G$. Then $t \neq 0$ in $K$ because $t = o(N)$. Therefore $I = \bigcap_{i=1}^{\ell} JK(P^i)K(G)$. Now $\hat{P}_i$ annihilates $JK(P_i)K(G)$ on the left so $\hat{P}_i I = 0 \forall i$. Since $\hat{G} = \hat{N}\hat{P}$ we have $\hat{G} I = 0$. Now $G = N \cup P_1 \cup P_2 \cup \cdots \cup P_i$ where unions are disjoint on the nonidentity elements. Therefore $\hat{G} + t \cdot 1 = \hat{N} + \sum_{i=1}^{\ell} \hat{P}_i$ in $K(G)$. But $\hat{P}_i I = 0 \forall i$ and $\hat{G} I = 0$. Therefore $(t \cdot 1 - \hat{N}) I = 0$. Let $\alpha \in I$. Then $(t \cdot 1 - \hat{N}) \alpha = 0$ and so $\alpha = \hat{N} \alpha / t$ (because $t \neq 0$ in $K$). Therefore $\alpha \in \hat{N} K(G)$. But $G = NP$ and $\hat{N} n = \hat{N} \forall n \in N$. Therefore $\alpha \in \hat{N} K(P)$. Let $\alpha = \hat{N} \beta$ where $\beta \in K(P)$ and $\beta = \sum_{x \in P} a_x y$. Since $\alpha \in I \subset w(K(G))$ we have $t \cdot (\sum_{x \in P} a_x) = 0$ and so $\sum_{x \in P} a_x = 0$ (because $t \neq 0$ in $K$). Thus $\beta \in w(K(P)) = JK(P)$. Hence $\alpha \in \hat{N} JK(P)$. This proves Theorem 1.

**Corollary 1.** Let $G$ be a finite group and $p \mid o(G)$. Suppose there exists a normal subgroup $G_0$ of $G$ such that $p \mid (G : G_0)$ and $G_0$ is a Frobenius group with $P$ as a complement subgroup. Then $JK(G) = \bigcap_{x \in G} JK(P^x)K(G)$.

**Proof.** $JK(G) = JK(G_0)K(G)$ [5, Theorem 16.6] and by Theorem 1, $JK(G_0) = \bigcap_{x \in G} JK(P^x)K(G_0)$. Hence
\[ JK(G) = \left( \bigcap_{x \in G_0} JK(P^x)K(G_0) \right)K(G) \]
\[ = \bigcap_{x \in G_0} JK(P^x)K(G) = \bigcap_{x \in G} JK(P^x)K(G). \]

(The last equality follows from the fact that the Sylow \( p \)-subgroups of \( G \) are precisely those of \( G_0 \).)

Professor Passman in a letter asked if the converse of Corollary 1 holds. He thus asked: If \( G \) is a finite group such that \( p|o(G) \) and \( JK(G) = \bigcap JK(P)K(G) \) as \( P \) ranges over Sylow \( p \)-subgroups of \( G \) for some field \( K \) of characteristic \( p \neq 0 \), does there exist in \( G \) a normal subgroup \( G_0 \) such that \( p | (G : G_0) \) and that \( G_0 \) is a Frobenius group with a Sylow \( p \)-subgroup as a complement? We give below a generalization (Corollary 2) of Corollary 1. This generalization yields a negative answer to the above question.

**Corollary 2.** Suppose there exist subgroups \( G_0, P_0 \) of the finite group \( G \) such that \( G \supseteq G_0 \supseteq P \supseteq P_0, \ G_0 \triangleleft G, \ P_0 \triangleleft G_0 \) and \( G_0/P_0 \) is a Frobenius group with \( P/P_0 \) as a complement. Then \( JK(G) = \bigcap JK(P)K(G) \), \( P \) ranging over Sylow \( p \)-subgroups of \( G \).

**Proof.** Since \( G_0/P_0 \) is a Frobenius group with \( P/P_0 \) as a complement. By Theorem 1 we have
\[ JK(G_0/P_0) = \bigcap_{x \in G_0} w(K(P^x/P_0))K(G_0/P_0). \]
Taking the complete preimage of both sides under the canonical map from \( K(G_0) \) to \( K(G_0/P_0) \) we get
\[ JK(G_0) = \bigcap_{x \in G_0} JK(P^x)K(G_0). \]
\[ JK(G) = JK(G_0)K(G) [5, Theorem 16.6]. \]
Hence
\[ JK(G) = \left( \bigcap_{x \in G_0} JK(P^x)K(G_0) \right)K(G) \]
\[ = \bigcap_{x \in G_0} JK(P^x)K(G) = \bigcap_{x \in G} JK(P^x)K(G). \]

**Example.** For \( p = 2 \), \( S_4 \) satisfies the conditions of Corollary 2, by taking \( G_0 = S_4 \) and \( P_0 = V_4 \), but \( S_4 \) does not satisfy the conditions of Corollary 1. This answers the question of Passman. Of course, now one can ask: Is the converse of Corollary 2 true?

**3. Counterexample.** The equality \( JK(G) = \bigcap JK(P)K(G) \) where \( P \) ranges over \( p \)-Sylow subgroups of \( G \) is false in general, as the following example shows.
Example (Passman). Let

\[ N = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} | \alpha, \beta, \gamma \in \mathbb{Z}/3\mathbb{Z} \].

\(N\) is a group of order 27 and \( \forall x \in N, x^3 = 1 \). It is easy to see that \( N \) is generated by \( a, b, c \) where

\[ a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \]

\((a, b) = c \) and \( c \) is central. Define \( \sigma: N \to N \) by

\[ \sigma \left( \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -\alpha & \beta \\ 0 & 1 & -\gamma \\ 0 & 0 & 1 \end{pmatrix} . \]

One checks that \( \sigma \) is an automorphism of \( N \) of order 2 and \( \sigma(a) = a^2; \sigma(b) = b^2; \sigma(c) = c \). Let \( H = \langle x \rangle \) the cyclic group of order 2.

Let \( G = N \rtimes_e H \) the semidirect product of \( N \) by \( H \) i.e. \( xn = \sigma(n)x \ \forall n \in N \). \( G \) is generated by the elements \( x, a^{-1}xa \) and \( b^{-1}xb \). So \( G \) is generated by the elements of order 2. \( o(G) = 54 \) and \( H \) is a Sylow 2-subgroup of \( G \). Let \( K \) be an algebraically closed field of characteristic 2. We shall show that \( JK(G) \neq \cap_{x \in G} JK(H^x)K(G) \). Since \( G/N \approx H \) we have \( G' \subseteq N \). Also \( c = (a, b) \in G', xa = a^2x = a(ax) \) so \( a \in G' \), and \( xb = b^2x = b(bx) \) so \( b \in G' \). Thus \( G' = N \). Hence \( G = G'H \). Since \( c \in G - H \) and \( c \) is central we have \( N_G(H) \neq H \). So \( G'H \) is not a Frobenius group with \( H \) as a complement. Hence by [7, Theorem, p. 103] \( JK(G) \) is not central. But we shall show that \( \cap_{x \in G} JK(H^x)K(G) \) is central. Let \( I = \cap_{x \in G} JK(H^x)K(G) \). Now let \( x \) be an element of order 2 so that \( P = \langle x \rangle \) is a Sylow 2-subgroup of \( G \). Then

\[ I \subset JK(P)K(G) = (1 + x)K(G) \]

so \( (1 - x)I = 0 \). Since \( G \) is generated by elements of order 2 therefore \( w(K(G)) \) is generated as left ideal by the set \( \{(1 - x)|o(x) = 2\} \) [3, Lemma 1, p. 153]. Thus \( wK(G)I = 0 \) so \( I \subset \hat{G}K(G) \) [3, Lemma 2, p. 154]. Hence \( I \) is central.

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CENTRE FOR ADVANCED STUDY IN MATHEMATICS, PANJAB UNIVERSITY, CHANDIGARH-160014, INDIA