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ON FUGLEDE'S THEOREM AND OPERATOR TOPOLOGIES[1]

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ABSTRACT. For each normal operator N and neighborhood E of 0 in the strong or weak operator topology there is a neighborhood D of 0 in the same topology such that $\|B\| \leqslant 1$ and $NB - BN$ in D imply $N^*B - BN^*$ in E. An example is given to show that D depends on N as well as E.

A bounded linear operator N on a complex Hilbert space H is called normal in case $NN^* = N^*N$. One of the most useful results concerning normal operators is Fuglede's theorem [2], which states that any bounded linear operator B on H satisfying $BN = NB$ also satisfies $BN^* = N^*B$.

Moore [5], using techniques inspired by those of Rosenblum [6], proves an asymptotic version of Fuglede's theorem. The proof of Moore's theorem in [5] shows that for each $\varepsilon > 0$ there is a $\delta > 0$ (depending only on ε) such that if N is any normal operator with $\|N\| \leqslant 1$ and B is any operator with $\|B\| \leqslant 1$ and $\|NB - BN\| < \delta$, then $\|N^*B - BN^*\| < \varepsilon$. This result can be stated in an equivalent form: for each neighborhood E of 0 in the norm topology there is a neighborhood D of 0 in the same topology such that the conditions $\|N\| \leqslant 1$, $\|B\| \leqslant 1$ and $NB - BN \in D$ imply $N^*B - BN^* \in E$.

The present note examines assertions analogous to Moore's in which the norm topology is replaced by the strong or weak operator topology. We show that analogous results can be obtained, although here the neighborhood D depends not only on E but also on N. The techniques used in proving the following theorem and corollaries are based on those used in [5].

THEOREM. *If N is a normal operator and E is a neighborhood of* 0 *in the strong or weak operator topology, then there is a neighborhood D of* 0 *in the same topology such that the conditions $\|B\| \leqslant 1$ and $NB - BN \in D$ imply $N^*B - BN^* \in E$.*

PROOF. We assume, without loss of generality, that $\|N\| \leqslant 1$. Let f be a unit vector in H. Choose $\varepsilon > 0$, put $r = 6/\varepsilon$ and define the circle $\Gamma = \{\lambda: |\lambda| = r\}$. Let R be an integer such that $\sum_{s=R+1}^{\infty} (r^s/s!) < e^{-r}$.

We first prove the strong operator topology theorem; the proof of the weak operator topology theorem is similar.

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It suffices to consider the case $E = \{Y: \|Yf\| < \varepsilon\}$. Since the map $e(\lambda) = e^{i\lambda N^*}$ from Γ to the bounded linear operators on H is uniformly continuous, there exists a finite set $\Gamma_1 \subset \Gamma$ such that for each λ in Γ there is a γ in Γ_1 with $\|e(\lambda) - e(\gamma)\| + \|e(-\lambda) - e(-\gamma)\| < e^{-r}$. Put $\delta = (re^{2r})^{-1}$ and define $D = \{X: \|XN^t g\| < \delta$ for each $g = e^{-i(\bar{\gamma}N + \gamma N^*)}f$, γ in Γ_1, and $t = 0, 1, \ldots, R - 1\}$.

Now suppose B is any operator with $NB - BN = X \in D$ and $\|B\| \leqslant 1$; we shall show $N^*B - BN^* = Y \in E$.

The function ϕ defined by $\phi(\lambda) = e(\lambda)Be(-\lambda)f$ is a holomorphic vector-valued function of λ. Write $\|\phi\| = \max\{\|\phi(\lambda)\|: \lambda \text{ in } \Gamma\}$; we proceed to show $\|\phi\| < 5$. Let λ and γ be in Γ. Since $\|e(\lambda)\| \leqslant e^r$ for all λ in Γ and $\|B\| \leqslant 1$, we have the inequalities

$$\begin{aligned}\|\phi(\lambda) - \phi(\gamma)\| &\\ &\leqslant \|e(\lambda)Be(-\lambda) - e(\gamma)Be(-\lambda) + e(\gamma)Be(-\lambda) - e(\gamma)Be(-\gamma)\| \\ &\leqslant e^r\|e(\lambda) - e(\gamma)\| + e^r\|e(-\lambda) - e(-\gamma)\|.\end{aligned}$$

Using the definition of Γ_1, we conclude that $\|\phi\| \leqslant 1 + \max\{\|\phi(\gamma)\|: \gamma \text{ in } \Gamma_1\}$.

For each γ in Γ_1 we have

$$\begin{aligned}\|\phi(\gamma)\| &= \|e^{i\gamma N^*}Be^{-i\gamma N^*}f\| \\ &\leqslant \left\|\left(e^{i(\bar{\gamma}N + \gamma N^*)}Be^{-i(\bar{\gamma}N + \gamma N^*)} - e^{i\gamma N^*}Be^{-i\gamma N^*}\right)f\right\| + \left\|e^{i(\bar{\gamma}N + \gamma N^*)}Be^{-i(\bar{\gamma}N + \gamma N^*)}f\right\| \\ &\leqslant \left\|e^{i\bar{\gamma}N^*}\left(e^{i\bar{\gamma}N}B - Be^{i\bar{\gamma}N}\right)e^{-i(\bar{\gamma}N + \gamma N^*)}f\right\| + 1.\end{aligned}$$

We have used the fact that N is normal to add exponents. The operator $e^{-i(\bar{\gamma}N + \gamma N^*)}$ is a unitary operator, and we define the unit vector $g = e^{-i(\bar{\gamma}N + \gamma N^*)}f$. Thus we have $\|\phi(\gamma)\| \leqslant 1 + e^r\|(e^{i\bar{\gamma}N}B - Be^{i\bar{\gamma}N})g\|$.

In order to estimate $\|(e^{i\bar{\gamma}N}B - Be^{i\bar{\gamma}N})g\|$ we note that $N^sB - BN^s = \sum_{t=0}^{s-1} N^{s-1-t}(NB - BN)N^t$. Since $\|N\| \leqslant 1$ and $NB - BN \in D$, we have the inequality $\|(N^sB - BN^s)g\| \leqslant s\delta$. Using this inequality for $s = 0, 1, \ldots, R$, we get the estimate

$$\begin{aligned}\|(e^{i\bar{\gamma}N}B - Be^{i\bar{\gamma}N})g\| &\leqslant \sum_{s=0}^{R} (r^s/s!)(s\delta) + \sum_{R+1}^{\infty} \frac{r^s}{s!}\|N^sB - BN^s\| \\ &\leqslant r\sum_{s=1}^{R} \frac{r^{s-1}}{(s-1)!}(\delta) + \sum_{R+1}^{\infty} \frac{r^s}{s!}(2) \\ &< re^r(\delta) + 2e^{-r} = 3e^{-r}.\end{aligned}$$

Thus $\|\phi(\gamma)\| < 4$ for γ in Γ_1, and hence $\|\phi\| < 5$.

By Cauchy's integral formula [**4**, p. 96] it follows that

$$\phi'(0) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\lambda)}{\lambda^2}\, d\lambda.$$

Thus

$$\|\phi'(0)\| \leqslant \frac{2\pi r}{2\pi} \cdot \frac{1}{r^2} \cdot \|\phi\| < \frac{5}{r} < \varepsilon.$$

Notice that $\phi'(\lambda) = (iN^* e^{i\lambda N^*} B e^{-i\lambda N^*} - ie^{i\lambda N^*} BN^* e^{-i\lambda N^*})f$, and hence $\|\phi'(0)\| = \|(N^*B - BN^*)f\|$. This proves the theorem for the strong operator topology.

The proof of the weak operator topology theorem is almost the same. It suffices, by the polarization identity, to consider $E = \{Y\colon |\langle Yf,f\rangle| < \varepsilon\}$; define D by $D = \{X\colon |\langle N^{s-1-t}XN^t g, e^{-i\gamma N}f\rangle| < \delta$ for $s = 1, \ldots, R$ and $t = 0, \ldots, s-1$ and $g = e^{-i(\bar{\gamma}N + \gamma N^*)}$ for γ in $\Gamma_1\}$, where $\langle\ ,\ \rangle$ denotes the inner product on H. We define the holomorphic function ϕ by $\phi(\lambda) = \langle e(\lambda)Be(-\lambda)f,f\rangle$. All other definitions can be repeated verbatim.

Then $|\langle (N^sB - BN^s)g, e^{-i\gamma N}f\rangle| \leqslant s\delta$, and we get the estimate $|\langle (e^{i\bar{\gamma}N}B - Be^{i\bar{\gamma}N})g, e^{-i\gamma N}f\rangle| < 3e^{-r}$. From this we again obtain the inequality $\|\phi(\gamma)\| < 4$. Using this, we can show as before that $\|\phi\| < 5$. This and Cauchy's integral formula imply that $\|\phi'(0)\| < \varepsilon$, which is sufficient to prove the weak operator topology theorem.

COROLLARY 1. *If N_1 and N_2 are normal operators and if E is a neighborhood of* 0 *in the strong or weak operator topology, then there is a neighborhood D of* 0 *in the same topology such that the conditions $\|B\| \leqslant 1$ and $N_1B - BN_2 \in D$ imply $N_1^*B - BN_2^* \in E$.*

PROOF. Let H denote the space on which N_1 and N_2 act; define $\tilde{N}$ on $H \oplus H$ by

$$\tilde{N} = \begin{pmatrix} N_2 & 0 \\ 0 & N_1 \end{pmatrix}.$$

In the set of operators on $H \oplus H$ define the subset

$$\tilde{E} = \left\{ \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \colon Y_1, Y_2, Y_3, Y_4 \text{ in } E \right\};$$

it is easy to see that $\tilde{E}$ is a neighborhood of 0 on $H \oplus H$ in the same topology (strong or weak) that E is on H. By the Theorem, there is a neighborhood $\tilde{D}$ of 0 on $H \oplus H$ such that any operator $\tilde{B}$ on $H \oplus H$ with $\|\tilde{B}\| \leqslant 1$ and $\tilde{N}\tilde{B} - \tilde{B}\tilde{N}$ in $\tilde{D}$ has $\tilde{N}^*\tilde{B} - \tilde{B}\tilde{N}^*$ in E.

Define $D = \{X\colon \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right)$ is in $\tilde{D}\}$; this set D is a neighborhood of 0 on H in the same topology that $\tilde{D}$ is on $H \oplus H$. Suppose B is an operator on H with $\|B\| \leqslant 1$ and $N_1B - BN_2 = X$ in D. Put $\tilde{B} = \left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right)$. Then $\|\tilde{B}\| \leqslant 1$ and

$$\tilde{N}\tilde{B} - \tilde{B}\tilde{N} = \begin{pmatrix} 0 & 0 \\ N_1B - BN_2 & 0 \end{pmatrix} \text{ is in } \tilde{D}.$$

Hence

$$\tilde{N}^*\tilde{B} - \tilde{B}\tilde{N}^* = \begin{pmatrix} 0 & 0 \\ N_1^*B - BN_2^* & 0 \end{pmatrix} \text{ is in } \tilde{E},$$

which implies that $N_1^*B - BN_2^*$ is in E. This proves Corollary 1.

COROLLARY 2. *Let ψ be a complex-valued function continuous on the union of the spectra of the normal operators N_1 and N_2. For each neighborhood E of* 0 *in the strong or weak operator topology, there is a neighborhood D of* 0 *in the same topology such that the conditions $\|B\| \leqslant 1$ and $N_1B - BN_2 \in D$ imply $\psi(N_1)B - B\psi(N_2) \in E$.*

PROOF. The technique used to prove Corollary 1, together with the observation that $\psi(N_1 \oplus N_2) = \psi(N_1) \oplus \psi(N_2)$, shows that we can assume $N_1 = N_2 = N$. Let C denote the set of complex functions ψ, defined on $\sigma(N)$ (the spectrum of N) for which Corollary 2 holds.

We shall show that C includes the set of continuous functions on $\sigma(N)$. Constant functions are clearly in C, as is the function $\psi(\lambda) = \lambda$. The function $\psi(\lambda) = \bar{\lambda}$ is in C by the Theorem. It is easy to see that if ψ_1 and ψ_2 are in C, then $\alpha_1\psi_1 + \alpha_2\psi_2$ is in C for any complex numbers α_1 and α_2, and $\psi_1\psi_2$ is in C since

$$\begin{aligned}\psi_1(N)\psi_2(N)B &- B\psi_1(N)\psi_2(N) \\ &= \psi_1(N)\left[\psi_2(N)B - B\psi_2(N)\right] + \left[\psi_1(N)B - B\psi_1(N)\right]\psi_2(N).\end{aligned}$$

To show that C is closed in the supremum norm, let ψ_k be in C for $k = 1, 2, \ldots,$ and suppose ξ is a function such that $\sup\{|\xi(\lambda) - \psi_k(\lambda)|: \lambda \text{ in } \sigma(N)\} \leqslant 1/k$. Then for each k we have

$$\begin{aligned}\xi(N)B - B\xi(N) = \{\xi(N) - \psi_k(N)\}B &+ \{\psi_k(N)B - B\psi_k(N)\} \\ &+ B\{\psi_k(N) - \xi(N)\}.\end{aligned}$$

This shows ξ is in C since $\|B\| \leqslant 1$ and $\|\xi(N) - \psi_k(N)\| \leqslant 1/k$. By Weierstrass's Theorem, C includes the set of continuous functions on $\sigma(N)$. This proves Corollary 2.

The following example shows that the neighborhood D must depend on N as well as E, in contrast to Moore's result for the norm topology.

EXAMPLE. There is a sequence $\{N_k: k = 1, 2, \ldots\}$ of unitarily equivalent normal operators and an operator B such that $\{N_kB - BN_k\}$ converges to 0 in the strong operator topology, and $\{N_k^*B - BN_k^*\}$ converges to a nonzero operator in the weak operator topology.

To see this, we use a normal operator U, a subnormal operator T and an operator X such that $XT = UX$ and $XT^* \neq U^*X$ (for information about subnormal operators, see [3, §154]). One way to construct such operators is as follows.

Let $\{e_n: n = 0, \pm 1, \pm 2, \dots\}$ be an orthonormal basis of H. Let $\{w_n\}_{-\infty}^{\infty}$ be a sequence of positive numbers with $0 < w_n \leqslant w_{n+1}$ such that $0 \leqslant \lim_{n\to-\infty} w_n < 1 < \lim_{n\to+\infty} w_n < \infty$; define numbers $\beta(n)$ by

$$\beta(n) = w_0 \cdots w_{n-1} \qquad (n > 0),$$
$$\beta(0) = 1,$$
$$\beta(-n) = (w_{-1} \cdots w_{-n})^{-1} \qquad (n > 0).$$

Using these, we define the operator X by $Xe_n = \beta(n)^{-1}e_n$; since $\lim_{n\to\infty} \beta(n) = \infty = \lim_{n\to-\infty} \beta(n)$, it follows that X is bounded. Let T denote the (hyponormal) weighted shift operator $Te_n = w_n e_{n+1}$ and let U denote the unitary shift operator $Ue_n = e_{n+1}$. Then $UXe_n = \beta(n)^{-1}e_{n+1} = w_n\beta(n+1)^{-1}e_{n+1} = X(w_n e_{n+1}) = XTe_n$. This proves that $UX = XT$, and it is easy to see that $U^*X - XT^* \neq 0$.

The operator T is subnormal for certain sequences $\{w_n\}$ [7, p. 87]. In this case there is a sequence $\{M_k\}$ of normal operators, each unitarily equivalent to the minimal normal extension of T, such that $M_k \to T$ in the strong operator topology [1,p. 432, proof of Theorem 3.3]; hence $XM_k \to XT = UX$ in the strong operator topology. Also, $M_k^* \to T^*$ in the weak operator topology [3, Problem 90] and $XM_k^* \to XT^*$ in the weak operator topology.

Define the operators N_k and B on the space $H \oplus H$ by

$$N_k = \begin{pmatrix} M_k & 0 \\ 0 & U \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}.$$

Because

$$N_k B - BN_k = \begin{pmatrix} 0 & 0 \\ UX - XM_k & 0 \end{pmatrix},$$

the sequence $\{N_k B - BN_k\}$ converges to 0 in the strong operator topology. Analogously, $\{N_k^* B - BN_k^*\}$ converges in the weak operator topology to the operator $\left(\begin{smallmatrix} 0 & 0 \\ U^*X - XT^* & 0 \end{smallmatrix}\right)$, which is nonzero.

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