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ON THE NORM OF THE CANONICAL PROJECTION OF E^{***} ONTO $E^{\perp}$

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ABSTRACT. For each number t with $1 < t \leqslant 2$, a renorming of c_0 is given for which $\|I - P\| = t$, where P is the canonical projection of c_0^{***} on c_0^*.

If E is a Banach space, then its dual E^* resides canonically in the third dual E^{***}. The mapping P: $E^{***} \to E^*$ that restricts a functional in E^{***} to E defines a projection of E^{***} onto E^* with norm one. Thus, if E is nonreflexive of course, $1 \leqslant \|I - P\| \leqslant 2$. It is not hard to prove that if $E = c_0$, then $\|I - P\| = 1$. The paper [1] is devoted to proving $\|I - P\| = 2$ for $E = l_1$. Furthermore, it seems that $\|I - P\|$ is either 1 or 2 in the (very few) examples in which the problem seems tractable. Fran Sullivan thus posed the question whether there are examples for which $1 < \|I - P\| < 2$. We answer this by constructing, for each number t with $1 < t \leqslant 2$, a renorming of c_0 so that $\|I - P\| = t$.

We proceed as follows:

Let $0 < r \leqslant 1$. For $x = (x_1, x_2, \dots) \in c_0$, define

(*) $\|x\| = \max(r^{-1}|x_1|, \sup_n |x_1 - x_n|)$. Then we have the following:

(1) $(1 + r)^{-1}\|x\|_\infty \leqslant \|x\| \leqslant \max(2, r^{-1})\|x\|_\infty$, where $\|x\|_\infty = \sup_n |x_n|$.

(2) By virtue of (1), the dual of c_0 is l_1 and for $\lambda = (\lambda_1, \lambda_2, \dots) \in l_1$ the dual norm is

$$\|\lambda\| = r\left|\sum_{n=1}^{\infty} \lambda_n\right| + \sum_{n=2}^{\infty} |\lambda_n|.$$

(3) The second dual norm on l_∞ is given by the same formula as (*).

(4) If P is the canonical projection of l_∞^* onto l_1, then $\|I - P\| = 1 + r$.

PROOFS. (1) If $\|x\| \leqslant 1$, then $|x_1| \leqslant r$ and $|x_1 - x_n| \leqslant 1$. Hence, $|x_n| \leqslant 1 + r$ for $n \geqslant 1$. The other half is easier.

(2) Notice that for $x \in c_0$ we have

$$\sum_{n=1}^{\infty} \lambda_n x_n = \left(r \sum_{n=1}^{\infty} \lambda_n\right)\left(\frac{1}{r} x_1\right) + \sum_{n=2}^{\infty} \lambda_n (x_n - x_1)$$
$$\leqslant \left(r\left|\sum_{n=1}^{\infty} \lambda_n\right| + \sum_{n=2}^{\infty} |\lambda_n|\right)\|x\|.$$

Hence the dual of norm (*) on l_1 is no larger than the norm (2). To see that

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we get equality, assume (without loss of generality) $\sum_{n=1}^{\infty} \lambda_n \geqslant 0$. Given n, let $x_j = 0$ for $j > n$; $x_1 = r$; $x_j = r + 1$ if $1 < j \leqslant n$ and $\lambda_j > 0$; and $x_j = r - 1$ if $1 < j \leqslant n$ and $\lambda_j < 0$. Then

$$\sum_{j=1}^{\infty} \lambda_j x_j = r\left|\sum_{j=1}^{\infty} \lambda_j\right| + \sum_{j=2}^{\infty} |\lambda_j|.$$

Hence the norm in (2) is the dual of norm (*).

(3) This can be obtained using the same formula,

$$\sum_{n=1}^{\infty} \lambda_n x_n = \left(r \sum_{n=1}^{\infty} \lambda_n\right)\left(\frac{1}{r} x_1\right) + \sum_{n=2}^{\infty} \lambda_n (x_n - x_1),$$

as in (2). The details are omitted.

(4) We begin by defining Q_n: $l_\infty \to l_\infty$ by letting the first n coordinates of $Q_n x$ be zero and for $j > n$ let the jth coordinate be x_j. Now, $\|x\| \leqslant 1$ implies $|x_1| \leqslant r$ and $|x_j - x_1| \leqslant 1$, so $|x_j| \leqslant 1 + r$ for $j > 1$. Hence $\|Q_n x\|_\infty \leqslant 1 + r$. But if the first term of an element of l_∞ is zero, then its norm is the usual sup-norm. Thus $\|Q_n\| \leqslant 1 + r$.

Now we use the fact that l_∞^* can be considered as the space of Radon measures on the Stone-Čech compactification βN of the positive integers. (We use μ interchangeably as a measure and linear functional.) The canonical projection P of l_∞^* onto l_1 sends μ to the sequence $\{\mu\{n\}\}$ and so $(I - P)\mu$ is the restriction of μ to $\beta N \setminus N$. Let $\lambda = P\mu$ and $\tilde{\mu} = (I - P)\mu$. Since $\tilde{\mu}(N) = 0$, it follows that $\tilde{\mu}(Q_n x) = \tilde{\mu}(x)$ for each n and each $x \in l_\infty$. Thus

$$\begin{aligned} |\tilde{\mu}(x)| = |\tilde{\mu}(Q_n x)| &= |\mu(Q_n x) - \lambda(Q_n x)| \\ &\leqslant \|\mu\| \, \|Q_n x\| + \sum_{j=n}^{\infty} |\lambda_j x_j| \\ &\leqslant \|\mu\|(1 + r)\|x\| + \|x\|_\infty \sum_{j=n}^{\infty} |\lambda_j|. \end{aligned}$$

Taking the limit on n, we get $\|\tilde{\mu}\| \leqslant (1 + r)\|\mu\|$. Hence $\|I - P\| \leqslant 1 + r$.

Now, let $s \in \beta N \setminus N$. Let ε_s be the point mass at s and ε_1 be the point mass at $1 \in N$. Take a net $\{n_\gamma\}$ in N converging to s. If $\|x\| \leqslant 1$, then $|x_{n_\gamma}| \leqslant 1 + r$ for each $n_\gamma > 1$, so

$$|\varepsilon_s(x)| = \lim_\gamma |x_{n_\gamma}| \leqslant 1 + r.$$

Also $|x_{n_\gamma} - x_1| \leqslant 1$ for every γ and therefore

$$|(\varepsilon_s - \varepsilon_1)(x)| = \lim_\gamma |x_{n_\gamma} - x_1| \leqslant 1.$$

Thus $\|\varepsilon_s\| \leqslant 1 + r$ and $\|\varepsilon_s - \varepsilon_1\| \leqslant 1$. But if $y = (r, 1 + r, 1 + r, \ldots)$, then $\|y\| = 1$, $\varepsilon_s(y) = 1 + r$ and $(\varepsilon_s - \varepsilon_1)(y) = 1$. Hence $\|\varepsilon_s\| = 1 + r$, $\|\varepsilon_s - \varepsilon_1\| = 1$, and $(I - P)(\varepsilon_s - \varepsilon_1) = \varepsilon_s$ so $\|I - P\| = 1 + r$. This completes the proof.

A. L. Brown has very recently communicated to us another way of renorming c_0 to achieve the results given above as follows: Let T be an

isomorphism of c_0 onto the space of convergent sequences c. For each $t \in [0, 1]$, $\|x\|_t = t\|x\|_\infty + (1 - t)\|Tx\|_\infty$ defines an equivalent norm on c_0, and as t ranges from 0 to 1 $\|I - P\|$ ranges from 2 to 1.

The referee has pointed out that all renormings of c_0 presented here have duals isometric to l_1. Thus $\|I - P\|$ is not an isometric invariant of E^*. He also points out that examples isometric to these were introduced for another purpose in [2].

References

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