HYPERINVARIANT SUBSPACES OF $C_{11}$ CONTRACTIONS

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Abstract. For an operator $T$ on a Hilbert space let $\text{Hyperlat } T$ denote its hyperinvariant subspace lattice. Assume that $T$ is a completely nonunitary $C_{11}$ contraction with finite defect indices. In this note we characterize the elements of $\text{Hyperlat } T$ among invariant subspaces for $T$ in terms of their corresponding regular factorizations and show that elements in $\text{Hyperlat } T$ are exactly the spectral subspaces of $T$ defined by Sz.-Nagy and Foias. As a corollary, if $T_1, T_2$ are two such operators which are quasi-similar to each other, then $\text{Hyperlat } T_1$ is (lattice) isomorphic to $\text{Hyperlat } T_2$.

1. Introduction. Let $T$ be a bounded linear operator acting on a complex separable Hilbert space $H$. A subspace $K$ of $H$ is hyperinvariant for $T$ if $K$ is invariant for every operator that commutes with $T$. We denote by $\text{Hyperlat } T$ the lattice of all hyperinvariant subspaces of $T$. Recently several authors studied $\text{Hyperlat } T$ for certain classes of contractions. Uchiyama showed that $\text{Hyperlat } T$ is preserved, as a lattice, for quasi-similar $C_0(N)$ contractions and for completely injection-similar $C_0$ contractions with finite defect indices (cf. [6] and [7]). As a result he was able to determine $\text{Hyperlat } T$ indirectly for such contractions. Wu, in [8], determined $\text{Hyperlat } T$ when $T$ is a completely nonunitary (c.n.u.) contraction with a scalar-valued characteristic function or a direct sum of such contractions. In this note we investigate $\text{Hyperlat } T$ for c.n.u. $C_{11}$ contractions with finite defect indices. Our main result (Theorem 1) says that for such contractions elements in $\text{Hyperlat } T$ are exactly the spectral subspaces $H_F$ defined by Sz.-Nagy and Foias in [5]. Thus we can completely determine $\text{Hyperlat } T$ in terms of the well-known structure of the hyper-invariant subspace lattice of normal operators. As a corollary, we show that for such contractions $\text{Hyperlat } T$ is preserved, as a lattice, under quasi-similarities.

2. Preliminaries. A contraction $T$ is completely nonunitary (c.n.u.) if there exists no nontrivial reducing subspace on which $T$ is unitary. The defect indices of $T$ are, by definition,

$$d_T = \text{rank}(I - T^*T)^{\frac{1}{2}} \quad \text{and} \quad d_T^* = \text{rank}(I - TT^*)^{\frac{1}{2}}.$$
$T \in C_{11}$ (resp. $C_1$) if $T^n x \to 0$ (resp. $T^n x \to 0$) for all $x \neq 0$; $C_{11} = C_1 \cap C_1$. For a $C_{11}$ contraction $T$, $d_T = d_{\Theta_T}$. Let $\Theta_T$ denote the characteristic function of an arbitrary contraction $T$. There is a one-to-one correspondence between the invariant subspaces of $T$ and the regular factorizations of $\Theta_T$. If $K \subset H$ is invariant for $T$ with the corresponding regular factorization $\Theta_T = \Theta_2 \Theta_1$, then $T = [T_1 T_2]$ is the triangulation on $H = K \oplus K^\perp$, then the characteristic functions of $T_1$, $T_2$ are the purely contractive parts of $\Theta_1$, $\Theta_2$, respectively. For more details the readers are referred to [5].

For arbitrary operators $T_1$, $T_2$ on $H_1$, $H_2$, respectively, $T_1 < T_2$ denotes that there exists a one-to-one operator $X$ from $H_1$ onto a dense linear manifold of $H_2$ (called quasi-affinity) such that $XT_1 = T_2X$. $T_1$, $T_2$ are quasi-similar ($T_1 \sim T_2$) if $T_1 < T_2$ and $T_2 < T_1$. For any subset $E$ of the unit circle $C$, let $M_E$ denote the operator of multiplication by $e^{it}$ on the space $L^2(E)$ of square-integrable functions on $E$. It was proved in [9] that any c.n.u. $C_{11}$ contraction $T$ with finite defect indices is quasi-similar to a uniquely determined operator, called the Jordan model of $T$, of the form $M_{E_1} \oplus \cdots \oplus M_{E_k}$, where $E_1, \ldots, E_k$ are Borel subsets of $C$ satisfying $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_k$. In this case $E_1 = \{ t : \Theta_T(t) \text{ not isometric} \}$.

We use $t$ to denote the argument of a function defined on $C$. A statement involving $t$ is said to be true if it holds for almost all $t$ with respect to the Lebesgue measure. In particular, for $E, F \subset C, E = F$ means that $(E \setminus F) \cup (F \setminus E)$ has Lebesgue measure zero. For any subset $F$ of $C$, $F' = C \setminus F$.

3. Main results. We start with the following

**Lemma 1.** Let $T$ be a $C_{11}$ contraction on $H$ and $U$ be a unitary operator on $K$. If there exists a one-to-one operator $X: H \to K$ such that $XT = UX$, then $T$ is quasi-similar to the unitary operator $U|_{XH}$.

**Proof.** Since $T$, being a $C_{11}$ contraction, is quasi-similar to a unitary operator, the assertion follows from Lemma 4.1 of [2] immediately.

Let $T$ be a c.n.u. $C_{11}$ contraction on $H$ with finite defect indices and let $U = M_{E_1} \oplus \cdots \oplus M_{E_k}$ acting on $K = L^2(E_1) \oplus \cdots \oplus L^2(E_k)$ be its Jordan model. Let $X: H \to K$ and $Y: K \to H$ be quasi-affinities intertwining $T$ and $U$. For any Borel subset $F \subset E_1$, let

$$K_F = L^2(E_1 \cap F) \oplus \cdots \oplus L^2(E_k \cap F)$$

be the spectral subspace of $K$ associated with $F$. For the contraction $T$ we considered, $\sigma(T) \subset C$ holds and there has been developed a spectral decomposition (cf. [5, p. 318 and pp. 315–316, resp.]). Let $H_F$ denote the spectral subspace associated with $F \subset C$. Indeed, $H_F$ is the (unique) maximal subspace of $H$ satisfying (i) $TH_F \subset H_F$, (ii) $T_F = T|_{H_F} \in C_{11}$ and (iii) $\Theta_T(t)$ is isometric for $t$ in $F'$. Moreover $H_F$ is hyperinvariant for $T$. We shall show that such subspaces $H_F$ give all the elements in Hyperlat $T$. We prove this in a series of lemmas.
Lemma 2. For any Borel subset $F \subseteq E_1$, $\overline{XH_F} = K_F$.

Proof. Let $K_1 = \overline{XH_F}$. Since $T_F \equiv T|_{H_F}$ is of class $C_{11}$, Lemma 1 implies that $T_F$ is quasi-similar to the unitary operator $U|_{K_1}$. Consider $K$ as a subspace of $L^2_k$ in the natural way. Hence $K_1$ is a reducing subspace for the bilateral shift $M$ on $L^2_k$. From the well-known structure of reducing subspaces of $M$, we obtain that $K_1 = PL^2_k$, where $P$ is a measurable function from $C$ to the set of (orthogonal) projections on $C^k$. Since

$$K_1 \subseteq K = L^2(E_1) \oplus \cdots \oplus L^2(E_k),$$

we have

$$P(t)C^k \subseteq C^j \oplus 0 \oplus \cdots \oplus 0$$

for $t$ in $E_j \setminus E_{j+1}, j = 1, \ldots, k - 1$, and $P(t) = 0$ for $t$ in $E_1$. For almost all $t$, let $(\psi_j(t))_j$ be an orthonormal base of $C^k$ consisting of eigenvectors of $P(t)$, that is, such that

$$P(t)\psi_j(t) = \delta_j(t)\psi_j(t), \quad j = 1, \ldots, k,$$

where the eigenvalues $\delta_j(t)$ are arranged in nonincreasing order: $1 > \delta_1(t) > \cdots > \delta_k(t) > 0$ (cf. [5, p. 272]). Let

$$F_j = \{t : \text{rank } P(t) > j\} = \{t : \delta_j(t) > 0\} \quad \text{for } j = 1, \ldots, k.$$

Then $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k$, $E_j \supseteq F_j$ and $P(t)\psi_j(t) = \chi_{F_j}(t)\psi_j(t)$ for each $j$. Setting $x_j(t) = (v(t), \psi_j(t))$ for $v \in L^2_k$ where $(\cdot, \cdot)$ denotes the usual inner product in $C^k$, we have $v(t) = \sum_1^k x_j(t)\psi_j(t)$. Since for $v \in K_1$,

$$v(t) = P(t)v(t) = \sum_1^k \chi_{F_j}(t)x_j(t)\psi_j(t),$$

the induced transformation

$$v \rightarrow x_1x_{F_1} \oplus \cdots \oplus x_kx_{F_k}$$

maps $K_1$ isometrically onto $L^2(F_1) \oplus \cdots \oplus L^2(F_k)$ (cf. [5, p. 272]). Moreover $U|_{K_1}$ will be carried over by this transformation to $M_{F_1} \oplus \cdots \oplus M_{F_k}$. We infer that $F_1 = \{t : \Theta_{T_F}(t) \text{ not isometric}\} \subseteq F$ (cf. the remark in §2). Thus for $v \in K_1$, $v(t) = \sum_1^k \chi_{F_j}(t)x_j(t)\psi_j(t) = 0$ on $F'$, which shows that $v \in K_F$, and hence $K_1 \subseteq K_F$.

To show the other inclusion, let $x \in K_F$ and $K_2 = \overline{XH_{F'}}$. Since $H = H_F \setminus H_{F'}$, we have $K = K_1 \setminus K_2$. Hence there exist sequences $(y_n) \subseteq K_1$ and $(z_n) \subseteq K_2$ such that $y_n + z_n \rightarrow x$. From what we proved above, $(y_n) \subseteq K_F$ and $(z_n) \subseteq K_{F'}$. Since $K = K_F \oplus K_{F'}$, by applying the (orthogonal) projection onto $K_F$ on both sides of $y_n + z_n \rightarrow x$ we obtain $y_n \rightarrow x$. This shows that $x \in K_1$, completing the proof.

For any Borel subset $F \subseteq E_1$, let $q(K_F) = \overline{\bigvee_{ST=F} SYK_F}$. It is known that $q(K_F)$ is hyperinvariant for $T$ and $\overline{Xq(K_F)} = K_F$ (cf. [5, pp. 76–78]).
Lemma 3. For any Borel subset $F \subseteq E_1$, let $q(K_F)$ be defined as above. Then $q(K_F) = H_F$.

Proof. Let $\Theta_T = \Theta_2 \Theta_1$ be the regular factorization corresponding to $q(K_F)$. To complete the proof it suffices to show that (i) $\Theta_1$ is outer, (ii) $\Theta_1(t)$ is isometric for $t$ in $F'$ and (iii) $\Theta_2(t)$ is isometric for $t$ in $F$ (cf. [5, pp. 312 and 205]). Since $q(K_F) \in \text{Hyperlat } T$, $\sigma(T|_{q(K_F)}) \subseteq \sigma(T)$ (cf. [1, Lemma 3.1]). It follows that $T|_{q(K_F)}$ is also of class $C_{11}$ (cf. [5, p. 318]), and hence $\Theta_1$ is outer (from both sides). This proves (i).

Since $X q(K_F) = K_F$ and $Y K_F \subseteq q(K_F)$, on the decompositions $H = q(K_F) \oplus q(K_F)^\perp$ and $K = K_F \oplus K_F^\ast$, $X$, $Y$, $T$ and $U$ can be triangulated as

$X = \begin{bmatrix} X_1 & * \\ 0 & X_2 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 & * \\ 0 & Y_2 \end{bmatrix}$, $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$, $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$.

It is easily seen that $X_1$ is a quasi-affinity intertwining $T_1$, $U_1$, so that $T_1 \prec U_1$. Since $T_1 = T|_{q(K_F)}$ is a $C_{11}$ contraction, we conclude from Lemma 1 that $T_1 \sim U_1$. This shows that $U_1 = \Sigma_{j=1}^k \Theta M_{E_j \cap F}$ is the Jordan model of $T_1$, and hence $F = E_1 \cap F = \{t: \Theta_1(t) \text{ not isometric}\}$. Therefore $F' = \{t: \Theta_1(t) \text{ isometric}\}$, which proves (ii). On the other hand, $X_2^* F_2$ and $Y_2^* F_2$ are one-to-one operators intertwining $T_2^*$, $U_2^*$. Note that $T_2$ is also of class $C_{11}$. (This follows from the fact that $\Theta_2 \equiv 0$ and [5, p. 318].) Let $V$ be the unitary operator quasi-similar to $T_2$. We infer that there are one-to-one operators intertwining $V^*$, $U_2^*$. It follows from Lemma 4.1 of [2] that $V^*$ and $U_2^*$ are unitarily equivalent to direct summands of each other. By the third test problem in [4] we conclude that $V^*$, $U_2^*$, and hence $V$, $U_2$, are unitarily equivalent. So $T_2 \sim U_2$. A similar argument as above shows that $E_1 \cap F' = \{t: \Theta_2(t) \text{ not isometric}\}$. Hence $E_1' \cup F = \{t: \Theta_2(t) \text{ isometric}\}$, which proves (iii) and completes the proof.

Lemma 4. Let $\mathcal{M} \subseteq H$ be hyperinvariant for $T$ with the corresponding factorization $\Theta_T = \Theta_2 \Theta_1$ and let $F = \{t: \Theta_1(t) \text{ not isometric}\}$. Then $\mathcal{M} = H_F$.

Proof. As proved in Lemma 3, for hyperinvariant $\mathcal{M}$, $T|_{\mathcal{M}}$ is of class $C_{11}$. Since $\Theta_{T|_{\mathcal{M}}}(t)$ is isometric for $t$ in $F'$, the maximality of $H_F$ implies that $\mathcal{M} \subseteq H_F$; cf. the remark before Lemma 2. Hence $\mathcal{M} \subseteq H_F = K_F$, by Lemma 2. We claim that $K_F = \bigvee_{SU} \mathcal{M}$. Indeed, using Lemma 1 we can show that $T|_{\mathcal{M}}$ is quasi-similar to $U|_{\mathcal{M}}$. Now we proceed as in the proof of Lemma 2 with $\mathcal{M}$ in the role of $K_1$. Let $P$ be a projection-valued function defined on $C$ such that $x \mathcal{M} = PL^2_k$. Choose the orthonormal base $(\psi_j(t))_j^k$ of $C^k$ consisting of eigenvectors of $P(t)$, and let $F_j = \{t: \text{rank } P(t) > j\}$ for $j = 1, \ldots, k$. Note that for $v \in L^2_k$, $v = \sum_{j=1}^k x_j \psi_j$, where $x_j(t) = (v(t), \psi_j(t))$ for each $j$ and $v = \sum_{j=1}^k x_j \psi_j$ if $v \in \mathcal{M}$. As shown before, the transformation $v \rightarrow x_{F_j} x_j \Theta M_{F_j} x_k$ maps $\mathcal{M}$ isometrically onto $L^2(F_j) \oplus \cdots \oplus L^2(F_k)$, and hence $M_{F_j} \Theta M_{F_k}$ is the Jordan model of $T|_{\mathcal{M}}$. We have $F_1 = \{t: \Theta_{T|_{\mathcal{M}}}(t) \text{ not isometric}\} = F = E_1 \cap F$. For each $j$,
let $S_j$ be the operator on $K$ defined by

$$S_j(v) = 0 \oplus \ldots \oplus \chi_{E_j \cap F} x_i \oplus \ldots \oplus 0$$

for $v = \sum_{j=1}^{k} x_j \psi_j \in K$. It is easily seen that $S_j U = U S_j$ and

$$S_j X \mathcal{M} = 0 \oplus \ldots \oplus L^2(E_j \cap F) \oplus \ldots \oplus 0$$

for each $j$. It follows that $K_F = \bigvee_{SU = US} SX \mathcal{M}$, as asserted. By Lemma 3,

$$H_F = q(K_F) = \bigvee_{VT = TV} VY K_F = \bigvee_{VT = TV} SU \bigvee_{US} V Y S X \mathcal{M}.$$

Since $V Y S X$ commutes with $T$ and $\mathcal{M}$ is hyperinvariant for $T$, we have $H_F \subseteq \mathcal{M}$. This, together with $\mathcal{M} \subseteq H_F$, completes the proof.

Now we have the following main theorem.

**THEOREM 1.** Let $T$ be a c.n.u. $C_{11}$ contraction on $H$ with $d_T = d_T^* = n < \infty$. Let $K \subseteq H$ be an invariant subspace with the corresponding regular factorization $\Theta_T = \Theta_2 \Theta_1$ and let $E = \{t; \Theta_T(t) \text{ not isometric}\}$. Then the following are equivalent:

1. $K \in \text{Hyperlat } T$;
2. $K = H_F$ for some Borel subset $F \subseteq E$;
3. the intermediate space of $\Theta_T = \Theta_2 \Theta_1$ is of dimension $n$ and for almost all $t$, either $\Theta_2(t)$ or $\Theta_1(t)$ is isometric.

**Proof.** (1) $\Rightarrow$ (2). That $K = H_F$, where $F = \{t; \Theta_2(t) \text{ not isometric}\}$, is proved in Lemma 4. It is a simple matter to check that $F \subseteq E$.

(2) $\Rightarrow$ (3). Since $T|_{H_F} \in C_{11}$, the intermediate space of $\Theta_T = \Theta_2 \Theta_1$ is of dimension $n$ (cf. [5, p. 192]). The rest is proved in [5, p. 312].

(3) $\Rightarrow$ (1). Since the intermediate space of $\Theta_T = \Theta_2 \Theta_1$ is of dimension $n$ and $\det \Theta_1 \equiv 0$ (otherwise $\det \Theta_2 \equiv 0$), we conclude that $T|_K$ is of class $C_{11}$ (cf. [5, p. 318]). Therefore, $\Theta_1$ is outer (from both sides). This, together with the other condition in (3), implies that $K = H_F$, where $F = \{t; \Theta_1(t) \text{ not isometric}\}$ (cf. [5, p. 312]). Thus $K \in \text{Hyperlat } T$.

**COROLLARY 1.** Let $T$ be as in Theorem 1 and let $U = M_{E_1} \oplus \ldots \oplus M_{E_k}$, acting on $K$, be its Jordan model. Then Hyperlat $T$ is (lattice) isomorphic to Hyperlat $U$. Moreover, if $X: H \rightarrow K$ and $Y: K \rightarrow H$ are quasi-affinities intertwining $T$, $U$, then the mapping $\mathcal{M} \rightarrow X \mathcal{M}$ implements the lattice isomorphism from Hyperlat $T$ onto Hyperlat $U$, and its inverse is given by $\mathcal{M} \rightarrow q(\mathcal{M}) = \bigvee_{ST = TS} SY \mathcal{M}$. In this case, $T|_{\mathcal{M}}$ and $U|_{X \mathcal{M}}$ are quasi-similar to each other.

**Proof.** The first assertion follows from Theorem 1, [5, pp. 315–316] and the well-known structure of Hyperlat $U$ [3]. The rest are immediate consequences of Lemmas 1, 2 and 3.

**COROLLARY 2.** Let $T_1$, $T_2$ be c.n.u. $C_{11}$ contractions with finite defect indices. If $T_1$ is quasi-similar to $T_2$, then Hyperlat $T_1$ is (lattice) isomorphic to Hyperlat $T_2$. 
**Corollary 3.** Let $T$ be a c.n.u. $C_{11}$ contraction with finite defect indices. If $K_1, K_2 \in \text{Hyperlat } T$ and $T|_{K_1}$ is quasi-similar to $T|_{K_2}$, then $K_1 = K_2$.

**Proof.** $T|_{K_1} \sim T|_{K_2}$ implies that they have the same Jordan model, say, $U = M_{E_1} \oplus \cdots \oplus M_{E_k}$. By Theorem 1, $K_1 = H_{E_1} = K_2$.

**Added in proof.** After submitting this paper, the author was notified that the main result here was independently obtained by R. I. Teodorescu (Factorisations régulières et sous-espaces hyperinvariants, to appear in Acta Sci. Math. (Szeged)) for arbitrary c.n.u. $C_{11}$ contractions. However the approaches are completely different.

**References**


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