THE CARDINALITY OF QUASICONFORMALLY NONEQUIVALENT TOPOLOGICAL 3-BALLS WITH FLAT BOUNDARIES IS $\aleph_0$

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Abstract. The theorem mentioned in the title is proved. During the course of the proof, the failure for $n = 3$ of the following 2-dimensional result will also be established: The boundary of a Jordan domain $D$ in $n$-space is a quasiconformal $(n-1)$-sphere if every quasiconformal self-mapping of $D$ can be extended to a quasiconformal self-mapping of the whole space.

1. Introduction. Let $\Sigma^{n-1}$ be a topological $(n-1)$-sphere imbedded in the $n$-sphere $S^n$. By the Jordan-Brouwer separation theorem, $\Sigma^{n-1}$ divides $S^n$ into two domains, $D_1$ and $D_2$, and is their common boundary. The set $\Sigma^{n-1}$ is collared if there is a neighborhood $U$ of $\Sigma^{n-1}$ and a homeomorphism $h$ of $U \cap D_1$ (or of $U \cap D_2$) into $S^n$ carrying $\Sigma^{n-1}$ onto the equator $S^{n-1}$ of $S^n$. The set $\Sigma^{n-1}$ is bicollared if $h$ is defined in all of $U$ and maps $\Sigma^{n-1}$ onto $S^{n-1}$. The set $\Sigma^{n-1}$ is flat if there is a homeomorphism $h$ of $S^n$ onto itself which carries $\Sigma^{n-1}$ onto $S^{n-1}$. By results of Brown [1], [2], every bicollared $\Sigma^{n-1}$ in $S^n$ is flat and a collared component of $S^n - \Sigma^{n-1}$ is a topological $n$-ball.

The set $\Sigma^{n-1}$ is said to be quasiconformally collared (resp. quasiconformally bicollared) if the homeomorphism $h$ above is quasiconformal. The image of $S^{n-1}$ under a quasiconformal mapping of $S^n$ is generally referred to as a quasiconformal sphere, rather than a quasiconformally flat sphere. Gehring [4] has established quasiconformal analogues to Brown's results. In particular, a quasiconformally collared component of $S^n - \Sigma^{n-1}$ is a quasiconformal $n$-ball. The other component of $S^n - \Sigma^{n-1}$ need not be a quasiconformal $n$-ball, even in the case that $\Sigma^{n-1}$ is flat.

Let $\mathcal{F}$ be the collection of all topological $n$-balls in $S^n$ whose boundaries are flat $(n-1)$-spheres and whose exteriors are quasiconformal $n$-balls. We divide $\mathcal{F}$ into equivalence classes by regarding two domains in $\mathcal{F}$ as equivalent if they can be mapped quasiconformally onto each other. Let $E(\mathcal{F})$ denote the set of equivalence classes so obtained. We show that in 3-space $E(\mathcal{F})$ has the cardinality of a continuum. This stands in marked contrast with the situation in 2-space, where the corresponding cardinality is well known to be one. (For related questions, see Kopylov [6].) In the course of the proof, the failure for $n = 3$ of the following result, due to Rickman [8]
for \( n = 2 \), will also be established: The boundary of a Jordan domain \( D \) in \( n \)-space is a quasiconformal \((n-1)\)-sphere if every quasiconformal self-mapping of \( D \) can be extended to a quasiconformal self-mapping of the whole space.

2. Wedges. We consider domains \( D \) in \( \mathbb{R}^3 = \mathbb{R}^3 \cup \{ \infty \} \),

\[
D = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3: |x_2| < g(x_1), x_1 > 0 \},
\]

where the function \( g: [0, \infty) \to \mathbb{R}^1 \) satisfies the following conditions for some \( 0 < a < \infty \):

\[
\begin{align*}
(i) & \quad g \text{ is continuous, } g(0) = 0, g(u) > 0 \text{ for } u > 0, \\
& \quad \text{and } g(u) = g(a) \text{ for } u \geq a. \\
(ii) & \quad g' \text{ is continuous, bounded, and increasing in } (0, a). \\
(iii) & \quad \lim_{u \to 0} g'(u) = 0.
\end{align*}
\]

Such a domain \( D \) is called a wedge of angle zero. The union of the \( x_3 \)-axis and the point \( \infty \) is called the edge of \( D \). (The above terminology is taken from Gehring and Väisälä [5].) Obviously a wedge \( D \) is a Jordan domain whose boundary \( \partial D \) is a flat 2-sphere. The exterior of \( D \) is a quasiconformal 3-ball, while \( D \) is not. (See Gehring and Väisälä [5].) Hence \( \partial D \) is not a quasiconformal 2-sphere, i.e. \( \partial D \) is not quasiconformally bicollared.

We will show that no two of the wedges defined by the functions \( g(u) = u^p, p \in (1, \infty) \), can be mapped quasiconformally onto one another. For this we require an upper and a lower bound for the modulus \( M(\Gamma) \) of a certain path family \( \Gamma \). We let \( F(\Gamma) \) denote the set of all Borel-measurable extended real-valued functions \( \rho \) in \( \mathbb{R}^3 \) for which

\[
\int_{\gamma} \rho \, ds > 1
\]

for each locally rectifiable path \( \gamma \in \Gamma \). The modulus of \( \Gamma \) is defined as

\[
M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^3} \rho^3 \, dm.
\]

(For the theory of modulus and quasiconformal mappings, see Gehring [3] and Väisälä [10].)

**Lemma 1.** Let \( D \) be the wedge defined by the function \( g(u) = u^p \) \((p > 1)\), let \( r_0 > 0 \) be a number such that \( 0 < g'(r_0) < 1 \), for \( 0 < r < r_0 \) let

\[
Z(r) = \{ x = (x_1, x_2, x_3) : x_1^2 + x_3^2 < r \},
\]

and for \( 0 < r_1 < r_2 < r_0 \) let \( \Gamma(r_1, r_2) \) denote the family of all paths joining \( \partial Z(r_1) \) and \( \partial Z(r_2) \) in \( D \cap Z(r_2) - Z(r_1) \). Then
\[
\frac{A(p)}{(r_1^{1-p}/2 - r_2^{1-p}/2)^2} \leq M(\Gamma(r_1, r_2)) \leq \frac{2^{3/2}A(p)}{(r_1^{1-p}/2 - r_2^{1-p}/2)^2},
\]

where

\[
A(p) = 2^{-5/2}(p - 1)^2 \int_0^\pi (\sin \varphi)^p \, d\varphi.
\]

**Proof.** For the left-hand inequality, let \( p \in F(\Gamma(r_1, r_2)) \), let \((r, \varphi, x_2)\) be cylindrical coordinates in \(\mathbb{R}^3\) with the polar angle \(\varphi\) being measured from the positive half of the \(x_3\)-axis, and for \(r \in [r_1, r_2]\), \(\varphi \in (0, \pi), \nu \in (-1, 1)\) let

\[
\gamma_{\varphi\nu}(r) = (r, \varphi, \nu g(r \sin \varphi)).
\]

Since \(\gamma_{\varphi\nu}\) is a rectifiable path in \(\Gamma(r_1, r_2)\) and since \(g'(r_2) < 1\) by hypothesis and by (2), we obtain

\[
1 \leq \left( \int_{\gamma_{\varphi\nu}} \rho \, ds \right)^3 \leq \left( 2^{1/2} \int_{r_1}^{r_2} \rho \, dr \right)^3
\]

\[
\leq 2^{3/2} \int_{r_1}^{r_2} \rho^3 \, g(r \sin \varphi) \, dr \left( \int_{r_1}^{r_2} r^{-1/2} g(r \sin \varphi)^{-1/2} \, dr \right)^2
\]

by Hölder's inequality. Integrating with respect to \(\varphi\) and \(\nu\) yields

\[
\int_{\mathbb{R}^3} \rho^3 \, dm \geq \int_{-1}^{1} d\nu \int_0^\pi d\varphi \int_{r_1}^{r_2} \rho^3 \, g(r \sin \varphi) \, dr
\]

\[
\geq A(p)(r_1^{1-p}/2 - r_2^{1-p}/2)^{-2},
\]

where \(A(p)\) is as in (3). Since \(p \in F(\Gamma(r_1, r_2))\) was arbitrary, this gives the left-hand inequality.

The right-hand inequality is obtained by observing that

\[
\rho(x) = \begin{cases} 
  p - 1 & \text{if } x = (r, \varphi, x_2) \in D \cap Z(r_2) - Z(r_1), \\
  0 & \text{otherwise},
\end{cases}
\]

belongs to \(F(\Gamma(r_1, r_2))\).

We also need the following extension result which shows, in particular, that, contrary to the situation in the plane (Rickman [8]), the extendability of 3-dimensional quasiconformal mappings over a flat 2-sphere does not guarantee that the 2-sphere will be quasiconformal.

**Lemma 2.** All quasiconformal mappings between wedges can be extended to quasiconformal self-mappings of \(\mathbb{R}^3\).

**Proof.** This result was proved in [7].
Lemma 3. Let $D$ and $D^\ast$ be two wedges defined respectively by the functions $g(u) = u^p$ and $g^*(u) = u^{p^*}$, $p, p^* \in (1, \infty)$. Then $D$ can be mapped quasiconformally onto $D^\ast$ if and only if $p = p^*$. 

Proof. The sufficiency part is obvious. For the necessity part, suppose, for example, that $p < p^*$, and that, contrary to the assertion, there is a quasiconformal mapping $f$ of $D$ onto $D^\ast$. By Lemma 2, $f$ can be extended to a quasiconformal mapping of $\mathbb{R}^3$ onto itself. Denote this mapping again by $f$, let $E$ denote the common edge of $D$ and $D^\ast$, and for $x \in E - \{\infty, f(\infty)\}$ set

$$L(x, f^{-1}) = \limsup_{h \to 0} \frac{|f^{-1}(x + h) - f^{-1}(x)|}{|h|}.\$$

In the proof of Lemma 2 it is verified that $f(E) = E$. Utilizing an idea of Syčev [9], we note the existence of a point $x_0$ in $E - \{\infty, f(\infty)\}$ such that $L(x_0, f^{-1}) > 0$.

Otherwise $f^{-1}$ would be locally constant in $E - \{\infty, f(\infty)\}$. Assume, for convenience of notation, that $x_0 = 0 = f^{-1}(x_0)$. Let

$$L(r) = \max_{|x| = r} |f(x)|, \quad l(r) = \min_{|x| = r} |f(x)|,$$

$$H = \limsup_{r \to 0} \frac{L(r)}{l(r)}.$$

Since $H < \infty$ by the quasiconformality of $f$, there exist positive constants $r_0$ and $H_0$ with $g'(r_0) \in (0, 1]$ such that

$$L(r) / l(r) \leq H_0$$

whenever $r \in (0, r_0]$. Choose $r_0^*$ so that $g^*(r_0^*) \in (0, 1]$ and $D^\ast \cap Z(r_0^*)$ lies in $f(D \cap Z(r_0))$, where, for $r \in (0, 1]$, $Z(r)$ is as defined in Lemma 1. Next choose $c \in (0, L(0, f^{-1}))$ and let $(x_k)$ be a sequence of points in $D$ such that $x_k \to 0$, $|x_k| < r_0$, $|f(x_k)| < r_0^*$, and

$$|x_k| / |f(x_k)| > c.\$$

Denoting $|x_k| = r_k$ and using (4) and (5) we obtain

$$L(r_k) < H_0 l(r_k) \leq H_0 |f(x_k)| \leq H_0 r_k / c = C_0 r_k,$$

where $C_0 = H_0 / c$. Passing to a subsequence, we may assume that $C_0 r_k < r_0^*$ for every $k$. Since $g'(r_0) < 1$, it follows from (6) that $f(D \cap Z(r_k/2))$ lies in $D^\ast \cap Z(C_0 r_k)$. Let $\Gamma(r_k/2, r_0)$ be the family of all paths joining $\partial Z(r_k/2)$ and $\partial Z(r_0)$ in $D \cap Z(r_0) - Z(r_k/2)$ and let $\Gamma^*(C_0 r_k, r_0^*)$ be the family of all paths
joining $\partial Z(C_{0}r_{\delta})$ and $\partial Z(r_{\delta}^{*})$ in $D^{*} \cap Z(r_{0}) = Z(C_{0}r_{\delta})$. Since $f \Gamma(r_{k}/2, r_{0})$ is minorized by $\Gamma^{*}(C_{0}r_{k}^{*}, r_{0}^{*})$, we obtain

$$\frac{M(\Gamma(r_{k}/2, r_{0}))}{M(f \Gamma(r_{k}/2, r_{0}))} \geq \frac{M(\Gamma^{*}(C_{0}r_{k}^{*}, r_{0}^{*}))}{M(\Gamma^{*}(C_{0}r_{k}^{*}, r_{0}^{*}))} \geq 2^{-3/2} \frac{A(p)\left[(r_{k}/2)^{(1-p)/2} - r_{0}^{(1-p)/2}\right]}{A(p)\left[(C_{0}r_{k})^{(1-p^{*})/2} - r_{0}^{*}(1-p^{*})/2\right]}^{-2} \geq C^{*}r_{k}^{p-p^{*}}$$

by Lemma 1, where $C^{*}$ is a positive constant which does not depend on $k$. Letting $k \to \infty$ leads to a contradiction with the quasiconformality of $f$. The proof is complete.

3. Results. Since the boundary of a wedge is not a quasiconformal sphere, Lemma 2 yields:

**Theorem 1.** In 3-space there are Jordan domains $D$ whose boundaries are flat, but not quasiconformally flat, such that all quasiconformal self-mappings of $D$ can be extended to quasiconformal self-mappings of $\mathbb{R}^{3}$.

Since the cardinality of the collection of all subdomains of $\mathbb{R}^{3}$ is $c$, the cardinality of a continuum, Lemma 3 yields:

**Theorem 2.** In 3-space the cardinality of a maximal collection of quasiconformally nonequivalent Jordan domains with flat boundaries and quasiconformally collared exteriors is $c$.

**References**


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