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MEASURES WHOSE INTEGRAL TRANSFORMS ARE PLURIHARMONIC

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ABSTRACT. Kolaski proved an F. and M. Riesz type theorem for the unit ball in $\mathbf{C}^N$. This paper generalizes those results.

1.1. Introduction. Throughout this paper the complex field shall be denoted by $\mathbf{C}$, the nonnegative integers by Z_+ and the unit circle by T: $\mathbf{C}^N$, Z_+^N and T^N shall denote the Cartesian product of N copies of $\mathbf{C}$, Z_+ and T, respectively.

For $(z, w) \in \mathbf{C}^N \times \mathbf{C}^N$, $\alpha \in Z_+^N$ we define $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and $\alpha! = (\alpha_1!)(\alpha_2!) \cdots (\alpha_N!)$.

$\mathbf{C}^N$ shall be topologized by the inner product $\langle z, w\rangle = z\bar{w} + \cdots + z_N\bar{w}_N$; as is usual, we set $|z| = \langle z, z\rangle^{1/2}$.

The open unit ball in $\mathbf{C}^N$ is denoted by B; for $N = 1$, its boundary is T, for $N \geqslant 2$, its boundary is the sphere $S = \{|z| = 1\}$.

We shall let m_T, m_S and m_B denote Lebesgue measure on T, S and B, respectively, all normalized to have unit mass.

Given a (locally compact) subset E of $\mathbf{C}^N$, $M(E)$ shall denote the (Banach) space of bounded Borel measures on E.

A subset E of $\mathbf{C}^N$ shall be called T $[T^N]$ invariant providing $\omega z \in E$ whenever $\omega \in T$ $[T^N]$ and $z \in E$ (if $(\omega, z) \in T^N \times E$, then $\omega z = (\omega_1 z_1, \ldots, \omega_N z_N)$). A measure $\sigma \in M(E)$ is called T $[T^N]$ invariant providing $\sigma(\omega A) = \sigma(A)$ for all ω in T $[T^N]$ and all Borel subsets A of E. Equivalently, $\int f(\omega z)d\sigma(z) = \int f(z)d\sigma(z)$ for all ω in T $[T^N]$ and all f in $C_0(E)$.

A consideration of the group of holomorphic homeomorphisms of B, led to the discovery of the kernels β and χ defined by

$$\beta(z, w) = \left[(1 - \langle w, w\rangle)/|1 - \langle z, w\rangle|^2\right]^N$$

and

$$\chi(z, w) = \left[(1 - \langle w, w\rangle)/|1 - \langle z, w\rangle|^2\right]^{N+1},$$

for all $(z, w) \in \bar{B} \times B$.

With regards to the kernel β there is the following question. If $u \in M(S)$, if $\int_S \beta(z, \cdot)\, du(z)$ is pluriharmonic in B, and if $N \geqslant 2$, then do we have $u \ll m_S$?

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In [4] – [6], Forelli proved several results which strongly suggest that the answer is yes, but the question is still open.

In [11], it was shown that an analogous theorem holds for the kernel χ. This paper generalizes [11], our main result being given by Theorem 2.2.7 which states (essentially): If σ is a class K measure with $\sigma(\partial\Omega_\sigma) = 0$, if $u \in M(\Omega_\sigma)$, and if $\int_{\Omega_\sigma} K_\sigma(z, \cdot)\, du(z)$ is pluriharmonic in Ω_σ, then $u \ll \sigma$.

It is shown that K_{m_B} so that our Theorem 2.2.7 includes Theorem 2.3 of [11] as a special case. It is also shown that $K_{m_S} = \beta$, but Theorem 2.2.7 does not answer the question posed above because the condition $\sigma(\partial\Omega_\sigma) = 0$ fails when $\sigma = m_S$.

2.1. Class K measures.

DEFINITION 2.1.1. Given $\sigma \in M(\mathbf{C}^N)$ we set $F_\sigma = \text{Support } \sigma$ and $\Omega_\sigma =$ Interior Holomorphic Hull F_σ (i.e., $\bar{\Omega}_\sigma = \{z \in \mathbf{C}^N\colon |f(z)| \leqslant \operatorname{Sup}_{w \in F_\sigma}|f(w)|$, for all f entire$\}$). Moreover, for $(\alpha, \beta) \in Z_+^N \times Z_+^N$ we set $\hat{\sigma}(\alpha, \beta) = \int z^\alpha \bar{z}^\beta\, d\sigma(z)$.

DEFINITION 2.1.2. Let $\sigma \neq 0$ be a nonnegative, T^N-invariant measure with compact support in $\mathbf{C}^N$. If, in addition, $F_\sigma \not\subset \bigcup_{j=1}^N \{z_j = 0\}$ (i.e., F_σ is not contained in the union of the coordinate planes) then we shall call σ a class K measure.

The terminology is motivated by the fact that for each such measure, σ, there corresponds a kernel K_σ such that $K_\sigma \cdot \sigma$ reproduces functions pluriharmonic on Ω_σ (see Definition 2.2.1 and Theorem 2.2.5).

To gain some insight into the generality of these measures, we give an example of a class K measure σ which is mutually singular with respect to m_B even though its support is $\bar{B}$.

If $r_1, r_2, \ldots$ is an enumeration of the rationals in (0, 1), if $a_1, a_2, \ldots$ is a summable sequence of positive numbers, and if μ_n denotes normalized Lebesgue measure on the sphere $S_n = \{|z| = r_n\}$, then we set $\sigma = \sum_{n=1}^\infty a_n \cdot \mu_n$.

For the purpose of proving Lemma 2.1.4, we recall the following fact [9, Theorem 2.4.2].

THEOREM 2.1.3. *The convergence domain Ω of a power series $\sum a_\alpha z^\alpha$ is the interior of the set $\{z \in \mathbf{C}^N\colon \operatorname{Sup}_\alpha |a_\alpha z^\alpha| < \infty\}$. Moreover, the convergence is absolute and uniform on compact subsets of Ω so that the sum is analytic on Ω.*

LEMMA 2.1.4. *If σ is a class K measure, then the domain of convergence, Ω, of the power series $\sum_{\alpha \in Z_+^N}(z^{2\alpha}/\hat{\sigma}(\alpha, \alpha))$ is precisely Ω_σ.*

PROOF. To see that $\Omega_\sigma \subset \Omega$, let r be a point in F_σ. Let z be a point of $\mathbf{C}^N$ which is dominated by r in the sense that $|z_j| < |r_j|$ whenever $r_j \neq 0$. Let $\Lambda = \Lambda_Z$ denote the set of integers j for which $z_j = 0$. Then

$$\operatorname*{Sup}_\alpha \frac{|z^{2\alpha}|}{\hat{\sigma}(\alpha, \alpha)} = \operatorname*{Sup}_{\alpha \in Z_\Lambda} \frac{|z^{2\alpha}|}{\hat{\sigma}(\alpha, \alpha)}$$

where $Z_\Lambda = \{\alpha \in Z_+^N | \alpha_j = 0, \text{ for all } j \in \Lambda\}$.

If $\alpha \in Z_\Lambda$, then $\hat{\sigma}(\alpha, \alpha) > |z^{2\alpha}|\sigma(G_z \cap F_\sigma)$ where $G_z = \{v \in \mathbf{C}^N\colon |v_j| > |z_j|, \text{ for all } j \notin \Lambda\}$.

Since $r \in G_z \cap F_\sigma$, it follows from the above that

$$\operatorname*{Sup}_{\alpha} \frac{|z^{2\alpha}|}{\hat{\sigma}(\alpha, \alpha)} \leqslant \frac{1}{\sigma(G_z \cap F_\sigma)} < \infty.$$

Hence, by Theorem 2.1.3, $z \in \overline{\Omega}$ whenever z is dominated by some r in F_σ, consequently $F_\sigma \subset \overline{\Omega}$.

Since F_σ is a compact, T^N-invariant subset of $\mathbf{C}^N$, it follows from Lemma 1 of [**12**, p. 65] that the holomorphic hull of F_σ, namely $\overline{\Omega}_\sigma$, is the least log-convex Rinehardt set in $\mathbf{C}^N$ which contains F_σ.

Hence, $F_\sigma \subset \overline{\Omega}_\sigma \subset \overline{\Omega}$ and so $\Omega_\sigma \subset \Omega$.

Conversely, let $v \notin \overline{\Omega}_\sigma$.

Using Lemma 1 of [**12**, p. 65] once again, it is easily shown that there is an $\alpha \neq 0$ in Z_+^N and an $\varepsilon = \varepsilon_\alpha > 0$ such that $|v^\alpha| > (1+\varepsilon)^{|\alpha|} \cdot |z^\alpha|$ for all z in F_σ.

With α fixed, choose $z = z_\alpha$ in F_σ such that

$$|z^\alpha| = \operatorname*{Sup}_{u \in F_\sigma} \{|u^\alpha|\}.$$

Then

$$\frac{|v^{2n\alpha}|}{\hat{\sigma}(n\alpha, n\alpha)} > \frac{\left[(1+\varepsilon)^{|\alpha|} \cdot |z^\alpha|\right]^{2n}}{|z^\alpha|^{2n} \cdot \sigma(F_\sigma)} = \frac{(1+\varepsilon)^{2n|\alpha|}}{\sigma(F_\sigma)}$$

for all $n = 1, 2, \ldots$.

Hence, by Theorem 2.1.3, $v \notin \overline{\Omega}$ which completes the proof of Lemma 2.1.4.

THEOREM 2.1.5. *If σ is of class K and if A is a compact subset of Ω_σ, then the series $C_\sigma(z, w) = \sum_{\alpha \in Z_+^N} (\bar{z}^\alpha w^\alpha / \hat{\sigma}(\alpha, \alpha))$ converges absolutely and uniformly on $\overline{\Omega}_\sigma \times A$.*

PROOF. Given A a compact subset of Ω_σ, choose $\lambda > 1$ such that $\lambda A \subset \Omega_\sigma$.

If $(z, w) \in (\lambda\Omega_\sigma) \times ((1/\lambda)\Omega_\sigma)$, we have by Lemma 2.1.4 that

$$\operatorname*{Sup}_{\alpha} \left| \left[\hat{\sigma}(\alpha, \alpha)\right]^{-1} \left(\frac{1}{\lambda} z\right)^{2\alpha} \right| = b_z < \infty$$

and

$$\operatorname*{Sup}_{\alpha} \left| \left[\hat{\sigma}(\alpha, \alpha)\right]^{-1} (\lambda w)^{2\alpha} \right| = b_w < \infty.$$

Since

$$|z^\alpha \overline{w}^\alpha| = \left| \left(\frac{1}{\lambda} z\right)^\alpha \right| \cdot |(\lambda w)^\alpha| \leqslant \operatorname{Max}\left\{ \left| \left(\frac{1}{\lambda} z\right)^{2\alpha} \right|, |(\lambda w)^{2\alpha}| \right\}$$

it follows that

$$\operatorname*{Sup}_{\alpha}\left|\frac{z^{\alpha}\overline{w}^{\alpha}}{\hat{\sigma}(\alpha,\alpha)}\right| \leqslant \operatorname{Max}\{b_z, b_w\} < \infty.$$

An application of Theorem 2.1.3 completes the proof.

REMARKS. (a) Since σ is T^N-invariant, the functions $[\hat{\sigma}(\alpha,\alpha)]^{-1/2}\cdot z^{\alpha}$ ($\alpha \in Z_+^N$) form an orthonormal basis for a closed subspace $A^2(\sigma)$ of $L^2(\sigma)$. Consequently, $f(w) = \int f\cdot C_\sigma(\cdot, w)\, d\sigma$ ($f \in A^2(\sigma)$, $w \in \Omega_\sigma$) (the basic ideas on reproducing kernels are given in [2]).

(b) The kernels

$$C_{m_S}(z,w) = [1 - \langle w, z\rangle]^{-N} \quad \text{and} \quad C_{m_B}(z,w) = [1 - \langle w, z\rangle]^{-(N+1)}$$

are known as the Szegő and Bergman kernels for B, respectively. In [3], Fefferman gives a detailed analysis of the Bergman kernel. The Szegő kernel is considered in [1].

(c) If $\sigma_s = h^s\cdot m_B$, where $h^s(z) = (1-|z|^2)^s$ ($z \in B$, $s \in \mathbf{C}$, $\operatorname{Re} s > -1$) then

$$C_{\sigma_s}(z,w) = \binom{N+s}{N}[1 - \langle w, z\rangle]^{-(N+1+s)}.$$

(The derivation and an application to bounded projections can be found in [10].)

THEOREM 2.1.6. *Let σ be a class K measure satisfying $\sigma(\partial\Omega_\sigma) = 0$. If $f \in L^1(\sigma)$ is holomorphic in Ω_σ, then $f(w) = \int f\cdot C_\sigma(\cdot, w)\, d\sigma$ ($w \in \Omega_\sigma$).*

PROOF. For $0 < r < 1$, let f_r be a dilation of f, $f_r(w) = f(rw)$.

If f is holomorphic in Ω_σ, then

$$f_r(w) = \int f\cdot C_\sigma(\cdot, w)d\sigma \qquad (w \in \Omega_\sigma) \tag{2.1}$$

by remark (a).

If ν denotes Haar measure on the torus T^N, then [13, Theorem 3.4.2]

$$\int |f_r(\omega w)|\, d\nu(\omega) \leqslant \int |f(\omega w)|\, d\nu(\omega), \tag{2.2}$$

for $0 < r < 1$ and w in Ω_σ.

Integrating (2.2) (over Ω_σ) with respect to σ, applying Fubini's theorem and then using the T^N-invariance of σ, leads to $\int|f_r|\, d\sigma \leqslant \int|f|\, d\sigma$ ($0 < r < 1$).

We may now apply Lebesgue's dominated convergence theorem to (2.1) to complete the proof.

2.2. The kernels K_σ.

DEFINITION 2.2.1. Given σ, a class K measure, define K_σ: $\overline{\Omega}_\sigma \times \Omega_\sigma \to (0, \infty)$ by $K_\sigma(z,w) = |C_\sigma(z,w)|^2/C_\sigma(w,w)$.

Concerning the kernels K_σ, we note that $K_{m_B} = \chi$ and $K_{m_S} = \beta$ as a consequence of remark (b).

LEMMA 2.2.2. *If σ is a class K measure, then K_σ is continuous on $\bar{\Omega}_\sigma \times \Omega_\sigma$.*

PROOF. Follows directly from Theorem 2.1.5.

DEFINITION 2.2.3. Given a class K measure σ, with corresponding Ω_σ and K_σ, let $M(\bar{\Omega}_\sigma) \to^{T_\sigma} C^\infty(\Omega_\sigma)$ be given by $(T_\sigma\mu)(w) = \int K_\sigma(z, w)\, d\mu(z)$.

THEOREM 2.2.4. *The maps T_σ are one-to-one. That is, given a class K measure σ, the map T_σ: $M(\bar{\Omega}_\sigma) \to C^\infty(\Omega_\sigma)$ is one-to-one.*

PROOF. As the map T_σ is clearly linear, it suffices to show $T_\sigma\mu = 0$ implies $\mu = 0$.

For a fixed w in Ω_σ and μ in $M(\bar{\Omega}_\sigma)$, it follows from Definitions 2.2.1 and 2.2.3 and Theorem 2.1.5 that

$$(T_\sigma\mu)(w) = \left[C_\sigma(w, w)\right]^{-1} \sum_\alpha \sum_\beta \left[\hat{\sigma}(\alpha, \alpha)\hat{\sigma}(\beta, \beta)\right]^{-1} \left[\int z^\alpha \bar{z}^\beta\, d\mu\right] \bar{w}^\alpha w^\beta.$$

It is now clear that $T_\sigma\mu = 0$ implies $\int z^\alpha \bar{z}^\beta\, d\mu = 0$ for all $(\alpha, \beta) \in Z_+^N \times Z_+^N$. An application of the Stone-Weierstrass theorem completes the proof.

The following theorem is a direct consequence of Theorem 2.1.6 and Definition 2.2.1.

THEOREM 2.2.5. *Let σ be a class K measure satisfying $\sigma(\partial\Omega_\sigma) = 0$. If $f \in L^1(\sigma)$ is pluriharmonic in Ω_σ, then $f(w) = \int f \cdot K_\sigma(\cdot, w)\, d\sigma$ $(w \in \Omega_\sigma)$.*

In Lemma 2.2.6 (below), $\{K_s\}_{s=0}^\infty$ denotes the Féjer kernel for T and $\mu * K_s$ is the convolution of a measure μ with K_s (for the specific definitions, see [**11**, §1.1]).

LEMMA 2.2.6. *Let σ be of class K and let $\mu \in M(\bar{\Omega}_\sigma)$. If $T_\sigma\mu$ is pluriharmonic on Ω_σ, then $T_\sigma(\mu * K_s)$ is a pluriharmonic polynomial.*

PROOF. Replace χ: $\bar{B} \times B \to (0, \infty)$ by K_σ: $\bar{\Omega}_\sigma \times \Omega_\sigma \to (0, \infty)$ in the proof of Lemma 2.2 of [**11**].

THEOREM 2.2.7. *Let σ be of class K and let $\mu \in M(\bar{\Omega}_\sigma)$. If $\sigma(\partial\Omega_\sigma) = 0$ and if $T_\sigma\mu$ is pluriharmonic in Ω_σ, then $T_\sigma\mu \in L^1(\sigma)$ and $\mu = (T_\sigma\mu) \cdot \sigma$.*

PROOF. Having established Lemma 2.2.2, Theorems 2.2.4 and 2.2.5 and Lemma 2.2.6, the proof of Theorem 2.2.7 is completed by the argument given in the proof of Theorem 2.3 of [**11**].

3. The spaces $H(\sigma)$ and $\operatorname{Re} H(\sigma)$. Let $H(\sigma)$ [$\operatorname{Re} H(\sigma)$] denote the $w[M(\bar{\Omega}_\sigma), C(\bar{\Omega}_\sigma)]$ closure of the class of all measures in $M(\bar{\Omega}_\sigma)$ of the form $g\sigma$ [$(\operatorname{Re} g)\sigma$] where g is a polynomial (see [**11**, §3.1] for details).

The classical F. and M. Riesz theorem has been generalized as follows [**7**, Theorem 1.2].

THEOREM 3.1. *If $\mu \in H(\sigma)$, then $\mu \ll \sigma$.*

For class K measures, Theorem 3.1 can be phrased in terms of integral transforms.

THEOREM 3.2. *If σ is a class K measure, and if $T_\sigma\mu$ is holomorphic in Ω_σ, then $\mu \ll \sigma$.*

PROOF. If $T_\sigma\mu$ is holomorphic in Ω_σ, then $T_\sigma(\mu * K_s)$ is a polynomial (recall the proof of Lemma 2.2.6).

Since $\mu * K_s$ converges weakly to μ, it follows from Lemma 2.2.2 and Definition 2.2.3 that $\mu \in H(\sigma)$. Applying Theorem 3.1 completes the proof.

We shall now show that a theorem similar to 3.1 holds for the space $\operatorname{Re} H(\sigma)$.

THEOREM 3.3. *Let σ be a class K measure satisfying $\sigma(\partial\Omega_\sigma) = 0$ and let $f \in L^1(\sigma)$ be T-invariant and satisfy $|f| \geqslant \varepsilon > 0$, a.e. σ. If $\mu \in \operatorname{Re} H(\sigma)$, then $T_\sigma(\mu/f)$ is pluriharmonic on Ω_σ. Consequently, $\mu = T_\sigma(\mu/f)\cdot f\sigma$.*

PROOF. Having established Lemma 2.2.2 and Theorems 2.2.5 and 2.2.7, the proof of Theorem 3.2 is completed by the argument given in the proof of Theorem 3.3 of [**11**].

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