MEASURES WHOSE INTEGRAL TRANSFORMS ARE PLURIHARMONIC

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Abstract. Kolaski proved an F. and M. Riesz type theorem for the unit ball in \( \mathbb{C}^N \). This paper generalizes those results.

1.1. Introduction. Throughout this paper the complex field shall be denoted by \( \mathbb{C} \), the nonnegative integers by \( \mathbb{Z}_+ \) and the unit circle by \( T = \mathbb{C}^+ \). This paper shall denote the Cartesian product of \( N \) copies of \( \mathbb{C}, \mathbb{Z}_+ \) and \( T \), respectively.

For \( (z, w) \in \mathbb{C}^N \times \mathbb{C}^N \), \( \alpha \in \mathbb{Z}_+^N \) we define \( z^\alpha = z_1^{\alpha_1}z_2^{\alpha_2} \cdots z_N^{\alpha_N} \), \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N \) and \( \alpha! = (\alpha_1!)(\alpha_2!)(\alpha_3!)(\alpha_N!) \).

\( \mathbb{C}^N \) shall be topologized by the inner product \( \langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_N \bar{w}_N \); as usual, we set \( |z| = \langle z, z \rangle^{1/2} \).

The open unit ball in \( \mathbb{C}^N \) is denoted by \( B \); for \( N = 1 \), its boundary is \( T \), for \( N > 2 \), its boundary is the sphere \( S = \{|z| = 1\} \).

We shall let \( m_T, m_S \) and \( m_B \) denote Lebesgue measure on \( T, S \) and \( B \), respectively, all normalized to have unit mass.

Given a (locally compact) subset \( E \) of \( \mathbb{C}^N \), \( M(E) \) shall denote the (Banach) space of bounded Borel measures on \( E \).

A subset \( E \) of \( \mathbb{C}^N \) shall be called \( T \) [\( T^N \)] invariant providing \( \omega z \in E \) whenever \( \omega \in T \) [\( T^N \)] and \( z \in E \) (if \( (\omega, z) \in T^N \times E \), then \( \omega z = (\omega_1 z_1, \ldots, \omega_N z_N) \)). A measure \( \sigma \in M(E) \) is called \( T \) [\( T^N \)] invariant providing \( \sigma(\omega A) = \sigma(A) \) for all \( \omega \in T \) [\( T^N \)] and all Borel subsets \( A \) of \( E \). Equivalently, \( \int f(\omega z) d\sigma(z) = \int f(z) d\sigma(z) \) for all \( \omega \in T \) [\( T^N \)] and all \( f \in C_0(E) \).

A consideration of the group of holomorphic homeomorphisms of \( B \), led to the discovery of the kernels \( \beta \) and \( \chi \) defined by

\[
\beta(z, w) = \left[ \frac{1 - \langle w, w \rangle}{1 - \langle z, w \rangle} \right]^N
\]

and

\[
\chi(z, w) = \left[ \frac{1 - \langle w, w \rangle}{1 - \langle z, w \rangle} \right]^{N+1}
\]

for all \( (z, w) \in B \times B \).

With regards to the kernel \( \beta \) there is the following question. If \( u \in M(S) \), if \( \int_S \beta(z, \cdot) \ d\mu(z) \) is pluriharmonic in \( B \), and if \( N > 2 \), then do we have \( u \ll m_S \)?
In [4] – [6], Forelli proved several results which strongly suggest that the answer is yes, but the question is still open.

In [11], it was shown that an analogous theorem holds for the kernel \( \chi \). This paper generalizes [11], our main result being given by Theorem 2.2.7 which states (essentially): If \( \sigma \) is a class K measure with \( \sigma(\partial \Omega_o) = 0 \), if \( u \in M(\Omega_o) \), and if \( f_{\Omega_o} K_\sigma(z, \cdot ) \, du(z) \) is pluriharmonic in \( \Omega_o \), then \( u \ll \sigma \).

It is shown that \( K_m^o \) so that our Theorem 2.2.7 includes Theorem 2.3 of [11] as a special case. It is also shown that \( K_m^o = \beta \), but Theorem 2.2.7 does not answer the question posed above because the condition \( \sigma(\partial \Omega_o) = 0 \) fails when \( \sigma = m^o \).

2.1. Class K measures.

**Definition 2.1.1.** Given \( \sigma \in M(\mathbb{C}^N) \) we set \( F_\sigma = \text{Support } \sigma \) and \( \Omega_o = \text{Interior Holomorphic Hull } F_\sigma \) (i.e., \( \Omega_o = \{ z \in \mathbb{C}^N; |f(z)| < \sup_{w \in F_\sigma} |f(w)|, \text{ for all } f \text{ entire} \} \). Moreover, for \((a, \beta) \in \mathbb{Z}_+^N \times \mathbb{Z}_+^N \) we set \( \delta(a, \beta) = \int z_\alpha \beta \, do(z) \).

**Definition 2.1.2.** Let \( \sigma \neq 0 \) be a nonnegative, \( T_N \)-invariant measure with compact support in \( \mathbb{C}^N \). If, in addition, \( F_\sigma \subset \bigcup_{j=1}^N \{ z_j = 0 \} \) (i.e., \( F_\sigma \) is not contained in the union of the coordinate planes) then we shall call \( \sigma \) a class K measure.

The terminology is motivated by the fact that for each such measure, \( \sigma \), there corresponds a kernel \( K_\sigma \) such that \( K_\sigma \cdot \sigma \) reproduces functions pluriharmonic on \( \Omega_o \) (see Definition 2.2.1 and Theorem 2.2.5).

To gain some insight into the generality of these measures, we give an example of a class K measure \( \sigma \) which is mutually singular with respect to \( m_B \) even though its support is \( B \).

If \( r_1, r_2, \ldots \) is an enumeration of the rationals in (0, 1), if \( a_1, a_2, \ldots \) is a summable sequence of positive numbers, and if \( \mu_n \) denotes normalized Lebesgue measure on the sphere \( S_n = \{ |z| = r_n \} \), then we set \( \sigma = \sum_{n=1}^\infty a_n \cdot \mu_n \).

For the purpose of proving Lemma 2.1.4, we recall the following fact [9, Theorem 2.4.2].

**Theorem 2.1.3.** The convergence domain \( \Omega \) of a power series \( \sum a_\alpha z^\alpha \) is the interior of the set \( \{ z \in \mathbb{C}^N; \sup_\alpha |a_\alpha z^\alpha| < \infty \} \). Moreover, the convergence is absolute and uniform on compact subsets of \( \Omega \) so that the sum is analytic on \( \Omega \).

**Lemma 2.1.4.** If \( \sigma \) is a class K measure, then the domain of convergence, \( \Omega \), of the power series \( \sum_{\alpha \in \mathbb{Z}_+^N} (z^\alpha / \delta(\alpha, \alpha)) \) is precisely \( \Omega_o \).

**Proof.** To see that \( \Omega_o \subset \Omega \), let \( r \) be a point in \( F_\sigma \). Let \( z \) be a point of \( \mathbb{C}^N \) which is dominated by \( r \) in the sense that \( |z_j| < |r_j| \) whenever \( r_j \neq 0 \). Let \( \Lambda = \Lambda_z \) denote the set of integers \( j \) for which \( z_j = 0 \). Then

\[
\sup_{\alpha} \frac{|z^{2\alpha}|}{\delta(\alpha, \alpha)} = \sup_{\alpha \in \Lambda} \frac{|z^{2\alpha}|}{\delta(\alpha, \alpha)}
\]

where \( \Lambda = \{ \alpha \in \mathbb{Z}_+^N | a_j = 0, \text{ for all } j \in \Lambda \} \).
If $\alpha \in \mathbb{Z}_\Lambda$, then $\hat{\sigma}(\alpha, \alpha) > |z^{2\alpha}|\sigma(G_z \cap F_\sigma)$ where $G_z = \{v \in \mathbb{C}^N: |v_j| > |z_j|, \text{for all } j \notin \Lambda\}$.

Since $r \in G_z \cap F_\sigma$, it follows from the above that

$$\sup_{\alpha} \frac{|z^{2\alpha}|}{\hat{\sigma}(\alpha, \alpha)} < \frac{1}{\sigma(G_z \cap F_\sigma)} < \infty.$$ 

Hence, by Theorem 2.1.3, $z \in \overline{\Omega}$ whenever $z$ is dominated by some $r$ in $F_\sigma$, consequently $F_\sigma \subset \overline{\Omega}$.

Since $F_\sigma$ is a compact, $T^\sigma$-invariant subset of $\mathbb{C}^N$, it follows from Lemma 1 of [12, p. 65] that the holomorphic hull of $F_\sigma$, namely $\Omega_\sigma$, is the least log-convex Rinehardt set in $\mathbb{C}^N$ which contains $F_\sigma$.

Hence, $F_\sigma \subset \overline{\Omega_\sigma} \subset \overline{\Omega}$ and so $\Omega_\sigma \subset \Omega$.

Conversely, let $v \notin \overline{\Omega_\sigma}$.

Using Lemma 1 of [12, p. 65] once again, it is easily shown that there is an $\alpha \neq 0$ in $\mathbb{Z}_+^N$ and an $\varepsilon = \varepsilon_\sigma > 0$ such that $|v^\alpha| > (1 + \varepsilon)^{|\alpha|} |z^\alpha|$ for all $z$ in $F_\sigma$.

With $\alpha$ fixed, choose $z = z_\alpha$ in $F_\sigma$ such that $|z^\alpha| = \sup_{u \in F_\sigma} \{|u^\alpha|\}$.

Then

$$\frac{|v^{2\alpha}|}{\hat{\sigma}(n\alpha, n\alpha)} > \frac{\left[(1 + \varepsilon)^{|\alpha|} |z^\alpha|\right]^{2n}}{|z^\alpha|^{2n} \cdot \sigma(F_\sigma)} = \frac{(1 + \varepsilon)^{2n|\alpha|}}{\sigma(F_\sigma)}$$

for all $n = 1, 2, \ldots$.

Hence, by Theorem 2.1.3, $v \notin \overline{\Omega}$ which completes the proof of Lemma 2.1.4.

**Theorem 2.1.5.** If $\sigma$ is of class $K$ and if $A$ is a compact subset of $\Omega_\sigma$, then the series $C_\sigma(z, w) = \sum_{\alpha \in \mathbb{Z}_+^N}(\hat{z}^\alpha w^\alpha / \hat{\sigma}(\alpha, \alpha))$ converges absolutely and uniformly on $\lambda A \times A$.

**Proof.** Given $A$ a compact subset of $\Omega_\sigma$, choose $\lambda > 1$ such that $\lambda A \subset \Omega_\sigma$.

If $(z, w) \in (\lambda \Omega_\sigma) \times ((1/\lambda)\Omega_\sigma)$, we have by Lemma 2.1.4 that

$$\sup_{\alpha} \left[\left|\hat{\sigma}(\alpha, \alpha)\right|^{-1}\left(\frac{1}{\lambda} z\right)^{2\alpha}\right] = b_z < \infty$$

and

$$\sup_{\alpha} \left[\left|\hat{\sigma}(\alpha, \alpha)\right|^{-1}(\lambda w)^{2\alpha}\right] = b_w < \infty.$$ 

Since

$$|z^{\alpha \overline{w^\alpha}}| = \left|\left(\frac{1}{\lambda} z\right)^{\alpha} (\lambda w)^{\alpha}\right| \leq \max\left\{\left|\left(\frac{1}{\lambda} z\right)^{2\alpha}\right|, |(\lambda w)^{2\alpha}|\right\}$$
it follows that
\[
\sup_{\alpha} \left| \frac{z^{\alpha}w^{\alpha}}{\delta(\alpha, \alpha)} \right| < \max\{b_z, b_w\} < \infty.
\]

An application of Theorem 2.1.3 completes the proof.

**Remarks.** (a) Since $\sigma$ is $T^N$-invariant, the functions $[\delta(\alpha, \alpha)]^{-1/2} \cdot z^\alpha$ ($\alpha \in \mathbb{Z}^N_+$) form an orthonormal basis for a closed subspace $A^2(\sigma)$ of $L^2(\sigma)$. Consequently, $f(w) = \int f \cdot C_\sigma(\cdot, w) \, d\sigma$ ($f \in A^2(\sigma), w \in \Omega_\sigma$) (the basic ideas on reproducing kernels are given in [2]).

(b) The kernels
\[
C_{m_\sigma}(z, w) = [1 - \langle w, z \rangle]^{-N} \quad \text{and} \quad C_{m_\beta}(z, w) = [1 - \langle w, z \rangle]^{-(N+1)}
\]
are known as the Szegő and Bergman kernels for $B$, respectively. In [3], Fefferman gives a detailed analysis of the Bergman kernel. The Szegő kernel is considered in [1].

(c) If $\sigma_s = h^s \cdot m_B$, where $h^s(z) = (1 - |z|^2)^s$ ($z \in B, s \in \mathbb{C}, \text{Re } s > -1$) then
\[
C_{\sigma_s}(z, w) = \binom{N + s}{N} \left[1 - \langle w, z \rangle\right]^{-(N+1+s)}.
\]
(The derivation and an application to bounded projections can be found in [10].)

**Theorem 2.1.6.** Let $\sigma$ be a class K measure satisfying $\sigma(\partial \Omega_\sigma) = 0$. If $f \in L^1(\sigma)$ is holomorphic in $\Omega_\sigma$, then $f(w) = \int f \cdot C_\sigma(\cdot, w) \, d\sigma$ ($w \in \Omega_\sigma$).

**Proof.** For $0 < r < 1$, let $f_r$ be a dilation of $f$, $f_r(w) = f(rw)$.

If $f$ is holomorphic in $\Omega_\sigma$, then
\[
f_r(w) = \int f \cdot C_\sigma(\cdot, w) \, d\sigma \quad (w \in \Omega_\sigma)
\]  
(2.1)

by remark (a).

If $\nu$ denotes Haar measure on the torus $T^N$, then [13, Theorem 3.4.2]
\[
\int |f_r(\omega w)| \, d\nu(\omega) \leq \int |f(\omega w)| \, d\nu(\omega),
\]
(2.2)
for $0 < r < 1$ and $w \in \Omega_\sigma$.

Integrating (2.2) (over $\Omega_\sigma$) with respect to $\sigma$, applying Fubini’s theorem and then using the $T^N$-invariance of $\sigma$, leads to $\int |f_r| \, d\sigma \leq \int |f| \, d\sigma$ ($0 < r < 1$).

We may now apply Lebesgue’s dominated convergence theorem to (2.1) to complete the proof.

**2.2. The kernels $K_\sigma$.**

**Definition 2.2.1.** Given $\sigma$, a class K measure, define $K_\sigma \colon \overline{\Omega_\sigma} \times \Omega_\sigma \rightarrow (0, \infty)$ by $K_\sigma(z, w) = |C_\sigma(z, w)|^2 / C_\sigma(w, w)$.

Concerning the kernels $K_\sigma$, we note that $K_{m_\sigma} = \chi$ and $K_{m_\beta} = \beta$ as a consequence of remark (b).
**Lemma 2.2.2.** If $\sigma$ is a class $K$ measure, then $K_\sigma$ is continuous on $\overline{\Omega}_0 \times \Omega_0$.

**Proof.** Follows directly from Theorem 2.1.5.

**Definition 2.2.3.** Given a class $K$ measure $\sigma$, with corresponding $\Omega_0$ and $K_\sigma$, let $M(\overline{\Omega}_0) \to T^\ast C^\infty(\Omega_0)$ be given by $(T_\sigma \mu)(w) = \int K_\sigma(z, w) \, d\mu(z)$.

**Theorem 2.2.4.** The maps $T_\sigma$ are one-to-one. That is, given a class $K$ measure $\sigma$, the map $T_\sigma: M(\overline{\Omega}_0) \to C^\infty(\Omega_0)$ is one-to-one.

**Proof.** As the map $T_\sigma$ is clearly linear, it suffices to show $T_\sigma \mu = 0$ implies $\mu = 0$.

For a fixed $w$ in $\Omega_0$ and $\mu$ in $M(\overline{\Omega}_0)$, it follows from Definitions 2.2.1 and 2.2.3 and Theorem 2.1.5 that

$$(T_\sigma \mu)(w) = [C_\sigma(w, w)]^{-1} \sum_\alpha \sum_\beta \left[ \delta(\alpha, \alpha) \delta(\beta, \beta) \right]^{-1} \left[ \int z^{\alpha \beta} \, d\mu \right] w^{\alpha} \overline{w}^\beta.$$

It is now clear that $T_\sigma \mu = 0$ implies $\int z^{\alpha \beta} \, d\mu = 0$ for all $(\alpha, \beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$. An application of the Stone-Weierstrass theorem completes the proof.

The following theorem is a direct consequence of Theorem 2.1.6 and Definition 2.2.1.

**Theorem 2.2.5.** Let $\sigma$ be a class $K$ measure satisfying $\sigma(\partial \Omega_0) = 0$. If $f \in L^1(\sigma)$ is pluriharmonic in $\Omega_0$, then $f(w) = \int f \cdot K_\sigma(\cdot, w) \, d\sigma$ ($w \in \Omega_0$).

In Lemma 2.2.6 (below), $(K_t)^{\infty}_{r=0}$ denotes the F{\text{e}}jer kernel for $T$ and $\mu \ast K_t$ is the convolution of a measure $\mu$ with $K_t$ (for the specific definitions, see [11, §1.1]).

**Lemma 2.2.6.** Let $\sigma$ be of class $K$ and let $\mu \in M(\overline{\Omega}_0)$. If $T_\sigma \mu$ is pluriharmonic on $\Omega_0$, then $T_\sigma(\mu \ast K_t)$ is a pluriharmonic polynomial.

**Proof.** Replace $x: B \times B \to (0, \infty)$ by $K_t: \overline{\Omega}_0 \times \Omega_0 \to (0, \infty)$ in the proof of Lemma 2.2 of [11].

**Theorem 2.2.7.** Let $\sigma$ be of class $K$ and let $\mu \in M(\overline{\Omega}_0)$. If $\sigma(\partial \Omega_0) = 0$ and if $T_\sigma \mu$ is pluriharmonic in $\Omega_0$, then $T_\sigma \mu \in L^1(\sigma)$ and $\mu = (T_\sigma \mu) \cdot \sigma$.

**Proof.** Having established Lemma 2.2.2, Theorems 2.2.4 and 2.2.5 and Lemma 2.2.6, the proof of Theorem 2.2.7 is completed by the argument given in the proof of Theorem 2.3 of [11].

3. **The spaces** $H(\sigma)$ and $\text{Re } H(\sigma)$. Let $H(\sigma)$ [Re $H(\sigma)$] denote the $w[M(\overline{\Omega}_0), C(\overline{\Omega}_0)]$ closure of the class of all measures in $M(\overline{\Omega}_0)$ of the form $g \sigma$ [(Re $g$)$\sigma$] where $g$ is a polynomial (see [11, §3.1] for details).

The classical F. and M. Riesz theorem has been generalized as follows [7, Theorem 1.2].

**Theorem 3.1.** If $\mu \in H(\sigma)$, then $\mu \ll \sigma$. 

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For class $K$ measures, Theorem 3.1 can be phrased in terms of integral transforms.

**Theorem 3.2.** If $\sigma$ is a class $K$ measure, and if $T_\sigma\mu$ is holomorphic in $\Omega_\sigma$, then $\mu \ll \sigma$.

**Proof.** If $T_\sigma\mu$ is holomorphic in $\Omega_\sigma$, then $T_\sigma(\mu * K_\sigma)$ is a polynomial (recall the proof of Lemma 2.2.6).

Since $\mu * K_\sigma$ converges weakly to $\mu$, it follows from Lemma 2.2.2 and Definition 2.2.3 that $\mu \in H(\sigma)$. Applying Theorem 3.1 completes the proof.

We shall now show that a theorem similar to 3.1 holds for the space $\text{Re } H(\sigma)$.

**Theorem 3.3.** Let $\sigma$ be a class $K$ measure satisfying $\sigma(\partial \Omega_\sigma) = 0$ and let $f \in L^1(\sigma)$ be $T$-invariant and satisfy $|f| > \epsilon > 0$, a.e. $\sigma$. If $\mu \in \text{Re } H(\sigma)$, then $T_\sigma(\mu / f)$ is pluriharmonic on $\Omega_\sigma$. Consequently, $\mu = T_\sigma(\mu / f) \cdot f_\sigma$.

**Proof.** Having established Lemma 2.2.2 and Theorems 2.2.5 and 2.2.7, the proof of Theorem 3.2 is completed by the argument given in the proof of Theorem 3.3 of [11].

**BIBLIOGRAPHY**


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