ON TWO QUESTIONS OF HALMOS
CONCERNING SUBSPACE LATTICES

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Abstract. An example is constructed of a nonreflexive pentagonal lattice of
subspaces. It follows that reflexivity is not invariant under lattice
isomorphism, even for finite lattices.

Following Halmos [2], [3] we say that a lattice $\mathcal{L}$ of subspaces of a Hilbert
space is reflexive if there is a collection of bounded linear operators whose
lattice of invariant subspaces is $\mathcal{L}$. The questions of Halmos [3] referred to in
the title are:

(i) Is every subspace lattice which is (isomorphic to) the pentagon reflexive?
(ii) If two finite subspace lattices are isomorphic and one is reflexive, must
the other be reflexive?

In this note we exhibit a nonreflexive pentagon, giving a negative answer to
the first question. Since Halmos [3] has given an example of a reflexive
pentagon, this shows that the second question also has a negative answer.

A nonreflexive pentagon can be described as follows. Let $K$ denote the
diagonal operator on $l^2$ defined by $K\{a_n\} = \{a_n/n\}$, and define $x = \{1/n\}$
and $y = \{1/n^{3/2}\}$. Now let the Hilbert space $\mathcal{K}$ be $l^2 \oplus l^2$
and let $\mathcal{L}$ consist of $\{0\}$, $l^2 \oplus \{0\}$, $\mathcal{G} = \{z \oplus Kz: z \in l^2\}$, $\mathcal{G}_0 = \{z \oplus (Kz + \alpha x + \beta y): z \in l^2,$
$\alpha \in C$, $\beta \in C\}$, and $\mathcal{K}$. We must check that $\mathcal{L}$ is a lattice with Hasse
diagram:

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\{0\}$};
  \node (b) at (1,1) {$\mathcal{G}$};
  \node (c) at (2,2) {$\mathcal{G}_0$};
  \node (d) at (3,3) {$\mathcal{K}$};
  \node (e) at (1,3) {$l^2 \oplus \{0\}$};

  \draw[-stealth] (a) edge (b);
  \draw[-stealth] (b) edge (c);
  \draw[-stealth] (c) edge (d);
  \draw[-stealth] (d) edge (e);
  \draw[-stealth] (e) edge (a);
\end{tikzpicture}
\end{center}

All the relations of the above diagram are immediate except that $\mathcal{G}_0 \cap (l^2 \oplus \{0\}) = \{0\}$. But if $Kz + \alpha x + \beta y = 0$ then $\alpha x + \beta y$ is in the range of $K$.
Hence $\sum_{n=1}^{\infty} n^2|\alpha/n + \beta/n^{3/2}|^2 < \infty$, so $\alpha = \beta = 0$. Thus $Kz = 0$, and $z = 0$. It follows that $\mathcal{G}_0 \cap (l^2 \oplus \{0\}) = \{0\}$.

Received by the editors September 1, 1978.


Key words and phrases. Reflexive lattice, invariant subspace.

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Theorem. The lattice $\mathcal{L}$ is not reflexive.

Proof. Let $\mathcal{O}$ denote the algebra of all operators leaving the elements of $\mathcal{L}$ invariant. It is apparent (cf. [2], [3]) that $\mathcal{L}$ is reflexive only if every invariant subspace of $\mathcal{O}$ is an element of $\mathcal{L}$. We show that $\mathcal{M} = \{z \oplus (Kz + \gamma y) : z \in l^2, \lambda \in \mathbb{C}, K \text{ and } \gamma \text{ as above}\}$, which is not an element of $\mathcal{L}$, is invariant under every operator in $\mathcal{O}$.

Let $T \in \mathcal{O}$. With respect to the decomposition $l^2 \oplus l^2$ of $\mathcal{H}$, $T$ has the form $T = (A B; C D)$. The invariance of $\mathcal{O}$ under $T$ yields $KA + KBK = DK$. For convenience let $C = D - KB$; then $KA = CK$. Since $T$ leaves $\mathcal{O}_0$ invariant, $T(0 \oplus y)$ is in $\mathcal{O}_0$ and so is $T(0 \oplus y) - (By \oplus KBy) = 0 \oplus Cy$. Thus $Cy = ax + By$ for some $a$ and $b$. Note that $CK = KA$ implies that $C$ leaves the range of $K$ invariant. Hence, by Corollary 2 of [6], $C$ also leaves the range $\mathcal{R}$ of $K^{1/2}$ invariant. Now $y = K^{1/2}x$ is in $\mathcal{R}$, hence so is $Cy$ and so is $ax$. But $x \not\in \mathcal{R}$. Therefore $a = 0$ and $Cy = By$.

The invariance of $\mathcal{M}$ under $T$ is now easily seen. For if $z \oplus (Kz + \gamma y) \in \mathcal{M}$, then

\[
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
\begin{pmatrix}
z \\
Kz + \gamma y
\end{pmatrix}
= \begin{pmatrix}
Az + BKz + \gamma By \\
(C + KB)(Kz + \gamma y)
\end{pmatrix}
= \begin{pmatrix}
Az + BKz + \gamma By \\
KAz + \gamma By + KBKz + \gamma KBy
\end{pmatrix}
= \begin{pmatrix}
u \\
Ku + \gamma By
\end{pmatrix},
\]

where $u = Az + BKz + \gamma By$. Hence $\mathcal{M}$ is invariant under $T$.

Remarks. (a) Instead of quoting Corollary 2 of [6] in the above proof, we could use the more general Proposition 5 of [1], once the proof of Proposition 5 is modified; see the remarks following Corollary 2 of [6].

(b) For finite lattices of subspaces of finite-dimensional spaces, the answer to question (ii) above is affirmative. A theorem of Johnson [5] implies that such a lattice is reflexive if and only if it is distributive.

(c) There are many known results concerning reflexive lattices: see [4] for other references to recent work.

References


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