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THE AHLFORS ESTIMATE

JOHN ERIK FORNAESS[1]

ABSTRACT. The Ahlfors estimate gives an upper bound on the growth of a complete Hermitian metric on the punctured unit disc, whose Gaussian curvature is bounded above by -1. A. Sommese has obtained certain lower bounds on the growth as well. We answer two questions concerning lower bounds, raised by Sommese.

The well-known Ahlfors estimate is a basic tool in transcendental algebraic geometry [1]–[3].

PROPOSITION 1 (AHLFORS ESTIMATE). *Let $ds^2 = h(z)|dz|^2$ be a complete Hermitian metric on the punctured disc $\Delta^* = \{z \in \mathbf{C}\colon\ 0 < |z| < 1\}$. If the Gaussian curvature is bounded above by a negative constant $-b$, then $h < C|z|^{-2}(\ln 1/|z|)^{-2}$ for a constant C dependent only on b.*

It is a natural question to ask whether this estimate can be reversed [3], [4]. In [4] Sommese studied the case $h(z) = h(|z|)$ and proved under this additional assumption that for any $\varepsilon > 0$ and any $r \in \langle 0, 1\rangle$ there exists a constant $C > 0$ such that $h(z) \geqslant C|z|^{\varepsilon-2}$ whenever $0 < |z| < r$.

Question 2 (*Sommese* [4]). *Assume $h(|z|)\,|dz|^2$ is a complete Hermitian metric on the punctured disc and suppose that the Gaussian curvature is bounded above by the negative constant $-b$. Does there exist for any $\varepsilon > 0$, $r \in \langle 0, 1\rangle$ a constant $C > 0$ such that*

$$h(|z|) > C|z|^{-2}(\ln 1/|z|)^{-2-\varepsilon}$$

whenever $0 < |z| < r$.

Our main result is that this last estimate fails in general.

PROPOSITION 3. *There exists a complete Hermitian metric on Δ^*, $h(|z|)\,|dz|^2$, with Gaussian curvature bounded above by -1, such that there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \Delta^*$, $z_n \to 0$ with $h(|z_n|) \leqslant |z_n|^{-2}(\ln 1/|z_n|)^{-n}$.*

This example suggests that the result of Sommese mentioned above is the best possible.

In [4], Sommese also asked what kind of lower bounds we have if $h(z)$ is not necessarily radial. We prove here:

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PROPOSITION 4. *There exists a complete Hermitian metric on* Δ^*, $h(z)|dz|^2$, *with Gaussian curvature bounded above by* -1 *such that* $\liminf_{z\to 0} h(z) = 0$.

We are deeply grateful to A. Sommese for having introduced us to this problem and for furnishing essential information.

1. The construction of the metric in Proposition 3 follows an inductive process. We will need the following lemma.

LEMMA 5. *Assume* $H(t)$: $(-\infty, 0) \to \mathbf{R}$ *is a smooth function such that* $H''(t) \geqslant \exp(H(t))$ *and* $\int_{-\infty}^{-1} \exp(H(t)/2)\,dt = \infty$. *Then* $h(z)\,|dz|^2$ *is a complete Hermitian metric on the punctured disc, where* $h(z) = \exp[H(\ln|z|) - 2\ln|z|]$. *The curvature of* $(h(z)/2)|dz|^2$ *is bounded above by* -1.

PROOF. To show completeness, it suffices to verify that $\int_0^{1/2} \sqrt{h(r)}\,dr = \infty$. This follows from the identity $\int_0^{1/2}\sqrt{h(r)}\,dr = \int_{-\infty}^{-\ln 2} \exp(H(t)/2)\,dt$.

The curvature condition is equivalent to the estimate $(\Delta \ln h/2)/2 \geqslant h/2$ which is equivalent to $H''(t) \geqslant \exp(H(t))$. Q.E.D.

To construct the metric we will inductively choose intervals $\{(a_n, b_n), (c_n, d_n)\}_{n=1}^{\infty}$ with $a_n < b_n < c_n < d_n < a_{n-1}\ \forall n$ and $d_1 = -1$ and $\lim_{n\to\infty} a_n = -\infty$. The function H will be constructed to have the following properties:

$(\mathrm{i})_n$ $\int_{c_n}^{d_n} \exp(H(t)/2)\,dt > 1$.

$(\mathrm{ii})_n$ $H(t) < -n\ln|t|$ for at least one point in (a_n, b_n).

$(\mathrm{iii})_n$ On (c_n, d_n), $H(t) = C_n - (2 + \varepsilon_n)\cdot\ln|t|$ for some constants $\varepsilon_n > 0$, C_n.

$(\mathrm{iv})_n$ On (a_n, b_n), $H(t) = C_n' - (n+2)\ln|t - \alpha_n|$ for some constants $\alpha_n > 0$, C_n'.

Let us first choose $a_1 < b_1 < c_1 < d_1$ and define H on $(a_1, 0)$ to satisfy $(\mathrm{i})_1$–$(\mathrm{iv})_1$. We choose $C_1 = 0$. Then $(\mathrm{i})_1$ reduces to $\int_{c_1}^{-1}|t|^{-1-\varepsilon_1/2}\,dt > 1$. Since $\int_{-\infty}^{-1} dt/|t| = \infty$, this can clearly be arranged by choosing $\varepsilon_1 > 0$ sufficiently small and $-c_1 > 0$ sufficiently large. Hence $H(t) = -(2+\varepsilon_1)\ln|t|$ on $(c_1, 0)$ can be assumed to satisfy $(\mathrm{i})_1$. Also $H''(t) = (2+\varepsilon_1)|t|^{-2} > e^H = |t|^{-2-\varepsilon_1}$ so we have the right curvature condition on (c_1, d_1). (Because of scaling it suffices to have a metric on $\{0 < |z| < 1/e\}$, corresponding to $t < d_1$.)

We observe that $(\mathrm{i})_1$ and $(\mathrm{iii})_1$ are still satisfied if we increase $|c_1|$ as we do below whenever needed.

To define (a_1, b_1) we solve at first the equations

$$-(2+\varepsilon_1)\ln|c_1 - 1| = C_1' - 3\ln|c_1 - 1 - \alpha_1|,$$
$$-(2+\varepsilon_1)|c_1 - 1|^{-1} = -3|c_1 - 1 - \alpha_1|^{-1}$$

to obtain $\alpha_1 = |c_1 - 1|(1-\varepsilon_1)(2+\varepsilon_1)^{-1}$ and $C_1' = \ln(3/(2+\varepsilon_1))^3\,|c_1 - 1|^{1-\varepsilon_1}$. With these values, $C_1' - 3\ln|t - \alpha_1|$ and $-(2+\varepsilon_1)\ln|t|$ have the same values and first derivatives at $t = c_1 - 1$. If we let $\tilde{H}(t) = C_1' - 3\ln|t - \alpha_1|$ we get that $\tilde{H}''(t)/\exp\tilde{H}(t) = 3|t - \alpha_1|((2+\varepsilon_1)/3)|c_1 - 1|^{\varepsilon_1 - 1}$. Hence if $|c_1|$

is sufficiently large, $\tilde{H}''(t) > \exp \tilde{H}(t)$ when $t < c_1$. We let $H(t)$ extend to the left of $c_1 - 2$ as $\tilde{H}(t)$ and between $c_1 - 2$ and c_1 we let $H(t)$ be a smoothing of $C_1' - 3\ln|t - \alpha_1|$, $t \leq c_1 - 1$, and $-(2 + \varepsilon_1)\ln|t|$, $t > c_1 - 1$. In this fashion, $H(t)$ can be made smooth and satisfy $H''(t) > \exp H(t)$. We choose $a_1 < b_1 < c_1 - 2$ such that $H(t) < -\ln|t|$ for some $t \in (a_1, b_1)$. This will make H satisfy (i)$_1$–(iv)$_1$.

Assume next that (i)$_1$–(iv)$_1$, . . . , (i)$_n$–(iv)$_n$ are all satisfied and H is defined on $(a_n, 0)$ and $H''(t)/\exp H(t) > 1$ on $(a_n, -1)$.

Let $d_{n+1} = a_n - 1$ and consider the equation

$$C_{n+1} - (2 + \varepsilon_{n+1})\ln|d_{n+1}| = C_n' - (n+2)\ln|d_{n+1} - \alpha_n|.$$

We may assume that $-a_n > 0$ is so large that $(2 + \varepsilon_{n+1})/|d_{n+1}| < (n + 2)/|d_{n+1} - \alpha_n|$ for all $0 < \varepsilon_{n+1} < \frac{1}{2}$ say. We will find C_{n+1} and $\varepsilon_{n+1} > 0$ such that (i)$_{n+1}$ can be satisfied, i.e.,

$$\int_{c_{n+1}}^{d_{n+1}} \left(e^{C_{n+1}/2}/|t|^{1+\varepsilon_{n+1}/2}\right) dt > 1.$$

This can be obtained by choosing $\varepsilon_{n+1} > 0$ sufficiently small and $-c_{n+1}$ sufficiently large and by choosing C_{n+1} to satisfy the above equation. The condition that $C_{n+1} - (2 + \varepsilon_{n+1})\ln|t|$ satisfies $H''/\exp H > 1$ reduces to $|d_{n+1} - \alpha_n|^{n+2} > e^{C_n'} \cdot |d_{n+1}|^2$ which may be assumed.

We define $H(t)$ on $(c_{n+1}, 0)$ by smoothing

$$\max\{C_{n+1} - (2 + \varepsilon_{n+1})\ln|t|, C_n' - (n+2)\cdot\ln|t - \alpha_n|\}$$

near d_{n+1} and otherwise letting $H(t)$ be the already defined H for $t > d_{n+1}$ and be $C_{n+1} - (2 + \varepsilon_{n+1})\ln|t|$ on (c_{n+1}, d_{n+1}).

Let $\alpha_{n+1} = |c_{n+1} - 1|(n + 1 - \varepsilon_{n+1})/(2 + \varepsilon_{n+1})$ and

$$C_{n+1}' = C_{n+1} + \ln\left(\frac{n+3}{2 + \varepsilon_{n+1}}\right)^{n+3} \cdot |c_{n+1} - 1|^{n+1-\varepsilon_{n+1}}.$$

Then $C_{n+1}' - (n+3)\ln|t - \alpha_{n+1}|$ and H have the same value and slope at $t = c_{n+1} - 1$. Moreover, the function $C_{n+1}' - (n+3)\ln|t - \alpha_{n+1}|$ satisfies the equation

$$H''/e^H \geq \frac{(2 + \varepsilon_{n+1})^{n+3}}{(n+3)^{n+2}} \cdot \frac{|c_{n+1} - 1 - \alpha_{n+1}|^{n+1}}{e^{C_{n+1}}|c_{n+1} - 1|^{n+1-\varepsilon_{n+1}}} > 1$$

if $-c_{n+1}$ is sufficiently large. Hence we can obtain a_{n+1}, b_{n+1} by the same argument as was used to find a_1, b_1. Therefore we can arrange for (i)$_{n+1}$–(iv)$_{n+1}$ to be satisfied and for H to satisfy $H'' >> \exp H$ on $(a_{n+1}, -1)$.

Completing the induction we obtain a smooth function H defined on $(-\infty, 0)$ which satisfies the curvature condition $H'' > \exp H$ on $(-\infty, -1)$ and the completeness condition

$$\int_{-\infty}^{-1} \exp(H/2)\,dt \geq \sum_{n=1}^{\infty} \int_{c_n}^{d_n} \exp(H/2)\,dt = \infty.$$

Furthermore, for some point $t_n \in (a_n, b_n)$, $H(t_n) < -n\ln|t_n|$.

As in Lemma 5, define the metric $h(z)|dz|^2$ by $h(z) = \exp[H(\ln|z|) - 2\ln|z|]$. Then $h/2$ is a complete Hermitian metric whose Gaussian curvature is bounded above by -1. Furthermore, if $z_n = e^{t_n}$, we obtain $h(z_n) < |z_n|^{-2}(\ln 1/|z_n|)^{-n}$.

This proves Proposition 3. Q.E.D.

2. In this section we prove Proposition 4. First, let $h_0(z) = 1/z\bar{z}\ln z\bar{z}$. We will modify h_0 by multiplying with a suitable infinite product. Consider $\phi_n = \ln(|z - 1/n|/2)$, $n = 2, 3, \ldots,$ and let ϕ_n^* be a subharmonic smoothing of ϕ_n which is sufficiently close to ϕ_n for the following estimates to hold.

We choose a sufficiently rapidly decreasing sequence $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and define $\psi_n = \exp(\varepsilon_n \phi_n^*)$ and let $h(z) = h_0(z)\prod_{n=2}^{\infty}\psi_n$.

Then $h(z)|dz|^2$ is a smooth Hermitian metric on the punctured unit disc. Moreover, $h(z)$ is complete since the distance from $-\frac{1}{2}$ to any point can be made to be at most 1 less than the same distance measured with h_0.

To compute the curvature, we have $\Delta \ln h = \Delta \ln h_0 + \sum_{n=2}^{\infty}\Delta \ln \psi_n = 4h_0 + \sum_{n=2}^{\infty}\varepsilon_n\Delta\phi_n^* \geqslant 4h_0 > 4h(z)$. Hence the curvature is bounded above by -4. On the other hand

$$h\left(\frac{1}{n}\right) = h_0\left(\frac{1}{n}\right)\prod_{k=2}^{\infty}\psi_k\left(\frac{1}{n}\right) < n^2(\ln n)^{-2}\cdot\exp\left(\varepsilon_n\phi_n^*\left(\frac{1}{n}\right)\right) < \frac{1}{n}$$

as we clearly can arrange, since $\phi_n(1/n) = -\infty$. Hence $h(1/n) \to 0$ as $n \to \infty$. Q.E.D.

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Department of Mathematics, Princeton University, Princeton, New Jersey 08540