THE AHLFORS ESTIMATE

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Abstract. The Ahlfors estimate gives an upper bound on the growth of a complete Hermitian metric on the punctured unit disc, whose Gaussian curvature is bounded above by \(-1\). A. Sommese has obtained certain lower bounds on the growth as well. We answer two questions concerning lower bounds, raised by Sommese.

The well-known Ahlfors estimate is a basic tool in transcendental algebraic geometry [1]-[3].

Proposition 1 (Ahlfors estimate). Let \(ds^2 = h(z) |dz|^2\) be a complete Hermitian metric on the punctured disc \(\Delta^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \}\). If the Gaussian curvature is bounded above by a negative constant \(-b\), then \(h < C |z|^{-2} (\ln 1/|z|)^{-2}\) for a constant \(C\) dependent only on \(b\).

It is a natural question to ask whether this estimate can be reversed [3], [4]. In [4] Sommese studied the case \(h(z) = h(|z|)\) and proved under this additional assumption that for any \(\epsilon > 0\) and any \(r \in (0, 1)\) there exists a constant \(C > 0\) such that \(h(z) > C |z|^2 \ln 1/|z|\) whenever \(0 < |z| < r\).

Question 2 (Sommese [4]). Assume \(h(|z|) |dz|^2\) is a complete Hermitian metric on the punctured disc and suppose that the Gaussian curvature is bounded above by the negative constant \(-b\). Does there exist for any \(\epsilon > 0\), \(r \in (0, 1)\) a constant \(C > 0\) such that
\[ h(|z|) > C |z|^{-2} (\ln 1/|z|)^{-2-\epsilon} \]
whenever \(0 < |z| < r\).

Our main result is that this last estimate fails in general.

Proposition 3. There exists a complete Hermitian metric on \(\Delta^*, h(|z|) |dz|^2\), with Gaussian curvature bounded above by \(-1\), such that there exists a sequence \(\{z_n\}_{n=1}^{\infty} \subset \Delta^*, z_n \to 0\) with \(h(|z_n|) \leq |z_n|^{-2} (\ln 1/|z_n|)^{-n}\).

This example suggests that the result of Sommese mentioned above is the best possible.

In [4], Sommese also asked what kind of lower bounds we have if \(h(z)\) is not necessarily radial. We prove here:

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Proposition 4. There exists a complete Hermitian metric on $\Delta^*$, $h(z)|dz|^2$, with Gaussian curvature bounded above by $-1$ such that $\lim\inf_{z \to 0} h(z) = 0$.

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1. The construction of the metric in Proposition 3 follows an inductive process. We will need the following lemma.

Lemma 5. Assume $H(t), (-\infty, 0) \to \mathbb{R}$ is a smooth function such that $H''(t) > \exp(H(t))$ and $\int_{-\infty}^{-1} \exp(H(t)/2) dt = \infty$. Then $h(z)|dz|^2$ is a complete Hermitian metric on the punctured disc, where $h(z) = \exp[H(\ln|z|) - 2\ln|z|]$. The curvature of $(h(z)/2)|dz|^2$ is bounded above by $-1$.

Proof. To show completeness, it suffices to verify that $\int_{-1}^{0} \sqrt{h(r)} \ dr = \infty$. This follows from the identity $\int_{-1}^{0} \sqrt{h(r)} \ dr = \int_{-\infty}^{-1} \exp(H(t)/2) dt$.

The curvature condition is equivalent to the estimate $(\Delta \ln h/2)/2 > h/2$ which is equivalent to $H''(t) > \exp(H(t))$. Q.E.D.

To construct the metric we will inductively choose intervals $\{(a_n, b_n), (c_n, d_n)\}_{n=1}^{\infty}$ with $a_n < b_n < c_n < d_n < a_{n-1}$ and $d_1 = -1$ and $\lim_{n \to \infty} a_n = -\infty$. The function $H$ will be constructed to have the following properties:

(i) $\int_{a_n}^{b_n} \exp(H(t)/2) dt > 1$.

(ii) $H(t) < -n \ln|t|$ for at least one point in $(a_n, b_n)$.

(iii) On $(c_n, d_n), H(t) = C_n - (2 + \varepsilon_n) \ln|t|$ for some constants $\varepsilon_n > 0$, $C_n$.

(iv) On $(a_n, b_n), H(t) = C_n' - (n + 2) \ln|t - a_n|$ for some constants $\alpha_n > 0$, $C_n'$.

Let us first choose $a_1 < b_1 < c_1 < d_1$ and define $H$ on $(a_1, 0)$ to satisfy (i)-(iv). We choose $C_1 = 0$. Then (i) reduces to $\int_{c_1}^{b_1} |t|^{-1-\varepsilon_1/2} dt > 1$. Since $\int_{-\infty}^{-1} dt/|t| = \infty$, this can clearly be arranged by choosing $\varepsilon_1 > 0$ sufficiently small and $-c_1 > 0$ sufficiently large. Hence $H(t) = -(2 + \varepsilon_1) \ln|t|$ on $(c_1, 0)$ can be assumed to satisfy (i). Also $H''(t) = (2 + \varepsilon_1)|t|^{-2} > e^H = |t|^{-2-\varepsilon_1}$, so we have the right curvature condition on $(c_1, d_1)$. (Because of scaling it suffices to have a metric on $0 < |z| < 1/e$, corresponding to $t < d_1$.)

We observe that (i), and (iii), are still satisfied if we increase $|c_1|$ as we do below whenever needed.

To define $(a_1, b_1)$ we solve at first the equations

$$(2 + \varepsilon_1) \ln|c_1 - 1| = C_1' - 3 \ln|c_1 - 1 - \alpha_1|,$$

$$(2 + \varepsilon_1)|c_1 - 1|^{-1} = -3|c_1 - 1 - \alpha_1|^{-1},$$

to obtain $\alpha_1 = |c_1 - 1|(1 - \varepsilon_1)(2 + \varepsilon_1)^{-1}$ and $C_1' = \ln[(3/(2 + \varepsilon_1))^3 |c_1 - 1|^{1-\varepsilon_1}]$. With these values, $C_1' - 3 \ln|t - \alpha_1|$ and $-(2 + \varepsilon_1)\ln|t|$ have the same values and first derivatives at $t = c_1 - 1$. If we let $\tilde{H}(t) = C_1' - 3 \ln|t - \alpha_1|$ we get that $\tilde{H}''(t)/\exp\tilde{H}(t) = 3[t - \alpha_1]/(2 + \varepsilon_1)|c_1 - 1|^{1-\varepsilon_1}$. Hence if $|c_1|$
is sufficiently large, $\tilde{H}''(t) > \exp \tilde{H}(t)$ when $t < c_1$. We let $H(t)$ extend to the left of $c_1 - 2$ as $\tilde{H}(t)$ and between $c_1 - 2$ and $c_1$ we let $H(t)$ be a smoothing of $C'_1 - 3 \ln |t - \alpha_1|$, $t < c_1 - 1$, and $-(2 + \varepsilon_1) \ln |t|$, $t > c_1 - 1$. In this fashion, $H(t)$ can be made smooth and satisfy $H''(t) > \exp H(t)$. We choose $a_1 < b_1 < c_1 - 2$ such that $H(t) < -\ln |t|$ for some $t \in (a_1, b_1)$. This will make $H$ satisfy (i)−(iv).

Assume next that (i)$_1$−(iv)$_1$, . . . , (i)$_n$−(iv)$_n$ are all satisfied and $H$ is defined on $(a_n, 0)$ and $H''(t)/\exp H(t) > 1$ on $(a_n, -1)$.

Let $d_{n+1} = a_n - 1$ and consider the equation
\[
C_{n+1} - (2 + \varepsilon_{n+1}) \ln |d_{n+1}| = C'_n - (n + 2) \ln |d_{n+1} - \alpha_n|.
\]
We may assume that $-a_n > 0$ is so large that $(2 + \varepsilon_{n+1})/|d_{n+1}| < (n + 2)/|d_{n+1} - \alpha_n|$ for all $0 < \varepsilon_{n+1} < 1/2$ say. We will find $C_{n+1}$ and $\varepsilon_{n+1} > 0$ such that (i)$_{n+1}$ can be satisfied, i.e.,
\[
\int_{c_{n+1}}^{d_{n+1}} (e^{\varepsilon_{n+1}/2} / |t|^{1 + \varepsilon_{n+1}/2}) \, dt > 1.
\]
This can be obtained by choosing $\varepsilon_{n+1} > 0$ sufficiently small and $-c_{n+1}$ sufficiently large and by choosing $C_{n+1}$ to satisfy the above equation. The condition that $C_{n+1} - (2 + \varepsilon_{n+1}) \ln |t|$ satisfies $H''/\exp H > 1$ reduces to $|d_{n+1} - \alpha_n|^n + 2 > e^{C_n} \cdot |d_{n+1}|^2$ which may be assumed.

We define $H(t)$ on $(c_{n+1}, 0)$ by smoothing
\[
\max \{ C_{n+1} - (2 + \varepsilon_{n+1}) \ln |t|, C'_n - (n + 2) \ln |t - \alpha_n| \}
\]
near $d_{n+1}$ and otherwise letting $H(t)$ be the already defined $H$ for $t > d_{n+1}$ and be $C_{n+1} - (2 + \varepsilon_{n+1}) \ln |t|$ on $(c_{n+1}, d_{n+1})$.

Let $\alpha_{n+1} = |c_{n+1} - 1|(n + 1 - \varepsilon_{n+1})/(2 + \varepsilon_{n+1})$ and
\[
C'_{n+1} = C_{n+1} + \ln \left( \frac{n + 3}{2 + \varepsilon_{n+1}} \right) \cdot |c_{n+1} - 1|^{n+1 - \varepsilon_{n+1}}.
\]
Then $C'_{n+1} - (n + 3) \ln |t - \alpha_{n+1}|$ and $H$ have the same value and slope at $t = c_{n+1} - 1$. Moreover, the function $C'_{n+1} - (n + 3) \ln |t - \alpha_{n+1}|$ satisfies the equation
\[
H''/e^H > \frac{(2 + \varepsilon_{n+1})^{n+3}}{(n + 3)^{n+2}} \cdot \frac{|c_{n+1} - 1 - \alpha_{n+1}|^{n+1}}{e^{C_{n+1}} |c_{n+1} - 1|^{n+1 - \varepsilon_{n+1}}} > 1
\]
if $-c_{n+1}$ is sufficiently large. Hence we can obtain $a_{n+1}$, $b_{n+1}$ by the same argument as was used to find $a_1$, $b_1$. Therefore we can arrange for (i)$_{n+1}$−(iv)$_{n+1}$ to be satisfied and for $H$ to satisfy $H'' > \exp H$ on $(a_{n+1}, -1)$.

Completing the induction we obtain a smooth function $H$ defined on $(-\infty, 0)$ which satisfies the curvature condition $H'' > \exp H$ on $(-\infty, -1)$ and the completeness condition
\[
\int_{-\infty}^{-1} \exp(H/2) \, dt > \sum_{n=1}^{\infty} \int_{c_n}^{d_n} \exp(H/2) \, dt = \infty.
\]
Furthermore, for some point $t_n \in (a_n, b_n)$, $H(t_n) < -n \ln |t_n|$.
As in Lemma 5, define the metric \( h(z)|dz|^2 \) by \( h(z) = \exp[H(\ln |z|) - 2 \ln |z|] \). Then \( h/2 \) is a complete Hermitian metric whose Gaussian curvature is bounded above by \(-1\). Furthermore, if \( z_n = e^{\phi_n} \), we obtain \( h(z_n) < |z_n|^{-2}(\ln 1/|z_n|)^{-n} \).

This proves Proposition 3. Q.E.D.

2. In this section we prove Proposition 4. First, let \( h_0(z) = 1/z \bar{z} \ln z \bar{z} \). We will modify \( h_0 \) by multiplying with a suitable infinite product. Consider \( \phi_n = \ln(|z - 1/n|/2) \), \( n = 2, 3, \ldots \), and let \( \phi_n^* \) be a subharmonic smoothing of \( \phi_n \), which is sufficiently close to \( \phi_n \) for the following estimates to hold.

We choose a sufficiently rapidly decreasing sequence \( e_n > 0 \), \( e_n \to 0 \), and define \( \psi_n = \exp(e_n \phi_n^*) \) and let \( h(z) = h_0(z) \prod_{n=2}^{\infty} \psi_n \).

Then \( h(z)|dz|^2 \) is a smooth Hermitian metric on the punctured unit disc. Moreover, \( h(z) \) is complete since the distance from \(-\frac{1}{2}\) to any point can be made to be at most 1 less than the same distance measured with \( h_0 \).

To compute the curvature, we have \( \Delta \ln h = \Delta \ln h_0 + \sum_{n=2}^{\infty} \Delta \ln \psi_n = 4h_0 + \sum_{n=2}^{\infty} e_n \Delta \phi_n^* > 4h_0 > 4h(z) \). Hence the curvature is bounded above by \(-4\). On the other hand

\[
h\left(\frac{1}{n}\right) = h_0\left(\frac{1}{n}\right) \prod_{k=2}^{\infty} \psi_k\left(\frac{1}{n}\right) < n^2(\ln n)^{-2} \cdot \exp\left(e_n \phi_n^*\left(\frac{1}{n}\right)\right) < \frac{1}{n}
\]

as we clearly can arrange, since \( \phi_n(1/n) = -\infty \). Hence \( h(1/n) \to 0 \) as \( n \to \infty \). Q.E.D.

REFERENCES


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