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HOMEOMORPHISMS OF A SURFACE WHICH ACT TRIVIALLY ON HOMOLOGY

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ABSTRACT. Let $\mathfrak{M}$ be the mapping class group of a surface of genus $g > 3$, and $\mathfrak{I}$ the subgroup of those classes acting trivially on homology. An infinite set of generators for $\mathfrak{I}$, involving three conjugacy classes, was obtained by Powell. In this paper we improve Powell's result to show that $\mathfrak{I}$ is generated by a single conjugacy class and that $[\mathfrak{M}, \mathfrak{I}] = \mathfrak{I}$.

I. Let $M = M_{g,1}$ be an orientable surface of genus $g \geqslant 3$ with one boundary component. (We shall frequently refer to the boundary curve as "the hole".) Let $\mathfrak{M} = \mathfrak{M}_{g,1}$ be its mapping class group (that is, homeomorphisms of M which are 1 on the boundary modulo homeomorphisms which are isotopic to 1 by an isotopy which is fixed on the boundary), and let $\mathfrak{I} = \mathfrak{I}_{g,1}$ be the mapping classes of $\mathfrak{M}$ which induce the identity map on the homology group $H_1(M, Z)$. The group $\mathfrak{I}$ is of specific interest to topologists for a number of reasons. For example, every hommology 3-sphere is obtained as a Heegaard decomposition with glueing map in $\mathfrak{I}$; more precise knowledge about $\mathfrak{I}$ could thus conceivably give some information about homology spheres. For the group-theoretically inclined, $\mathfrak{I}$ supports a number of interesting problems. For example, it is an open question as to whether it is finitely generated. At present, information concerning $\mathfrak{I}$ is scarce; the main references are given at the end.

Let α be any bounding simple closed curve (BSCC) in M. It has then a well defined *genus* $g(\alpha)$, namely, the genus of the surface it bounds. (In contrast to the case of a closed surface, α bounds only on one side; the other side contains the hole.) Consider the group $\mathfrak{T}_k \subset \mathfrak{I}$ generated by all twists T_α on BSCC's α of genus k; $\mathfrak{T}_k$ is clearly a normal subgroup of $\mathfrak{M}$, since the genus of α is invariant under any homeomorphism h of M, and $T_{h(\alpha)} = hT_\alpha h^{-1}$.

If α_1, α_2 are two disjoint, homologous SCC's with α_i *not* homologous to zero (we shall write "$\simeq$" for "homologous") then (α_1, α_2) also has a genus $g(\alpha_1, \alpha_2)$, since α_1, α_2 bound a piece of M. If we let $\mathfrak{W}_k$ be the group generated by all maps of the form $T_{\alpha_1}T_{\alpha_2}{}^{-1}$ with $g(\alpha_1, \alpha_2) = k$ then $\mathfrak{W}_k \subset \mathfrak{I}$ is also normal in $\mathfrak{M}$. We shall speak of such a map as "a generator of $\mathfrak{W}_k$"; likewise, if α is a BSCC and $g(\alpha) = k$, T_α will be called "a generator of $\mathfrak{T}_k$". Note that all generators of a given type are conjugate in $\mathfrak{M}$; this is just the same as saying, for example, that if α, β are BSCC's of the same genus, then

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there is a homeomorphism h such that $\beta = h(\alpha)$.

Jerry Powell has shown in **[P]** that, for a *closed* surface of genus $g \geqslant 3$, $\mathcal{I}_g$ is generated by the generators of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{W}_1$, that is, $\mathcal{I}_g = \mathcal{W}_1 \cdot \mathcal{T}_1 \cdot \mathcal{T}_2$. Here $\mathcal{I}_g$ is the closed surface version of $\mathcal{I}_{g,1}$, while $\mathcal{T}_k$ has generators T_α with α bounding genus k on one side or the other (thus $\mathcal{T}_k = \mathcal{T}_{g-k}$ for a closed surface), and likewise for $\mathcal{W}_k$. We shall use Powell's result to produce a stronger one, namely that $\mathcal{I} = \mathcal{W}_1$, this result holding for both closed surfaces and surfaces with one hole. We will then use this to show that $[\mathfrak{M}, \mathcal{I}] = \mathcal{I}$.

For $g = 1, 2$ the above results fail. The group $\mathcal{I}_1$ is trivial (this is because $\pi_1(M_1)$ is abelian) and a theorem of Nielsen (see **[MKS**, Theorem 3.9, p. 165]) implies that $\mathcal{I}_{1,1}$ is infinite cyclic and generated by a twist on the boundary. For a closed surface of genus 2, $\mathcal{W}_1$ and $\mathcal{T}_2$ are clearly trivial and Powell shows that $\mathcal{I} = \mathcal{T}_1$. Using the methods of this paper, Powell's result also implies fairly easily that $\mathcal{I}_{2,1} = \mathcal{W}_1 \cdot \mathcal{T}_1$ and $\mathcal{T}_2 \subset \mathcal{W}_1$. We also find that $[\mathfrak{M}, \mathcal{I}] \neq \mathcal{I}$ for $g = 2$. The author has shown by different methods (unpublished) that $\mathcal{I}_2/[\mathfrak{M}_2, \mathcal{I}_2]$ (which must be cyclic) has Z_{10} as a quotient and that the corresponding group for $\mathcal{I}_{2,1}$ has $Z \oplus Z_2$ as a quotient. In the remainder of the paper we will assume $g \geqslant 3$.[1]

II. If $\mathfrak{M}_{g,1}$ is the mapping class group of a surface with one hole and $\mathfrak{M}_g$ that of a closed surface, there is a natural surjection $\mathfrak{M}_{g,1} \xrightarrow{p} \mathfrak{M}_g$ obtained by "filling in the hole". The kernel of p is generated by "moving the hole around"; precisely, by:

(a) twisting the hole,

(b) maps of type $T_{\alpha_1} T_{\alpha_2}^{-1}$ with α_1, α_2 disjoint, homologous, and $g(\alpha_1, \alpha_2) = g - 1$: see Figure 1. The map illustrated there has the effect of sliding the hole around the second handle. Note that twisting the hole itself is just T_α for $g(\alpha) = g$. Also, these maps are all in $\mathcal{I}_{g,1}$; hence we get an exact sequence: $0 \to \operatorname{Ker} p \to \mathcal{I}_{g,1} \to \mathcal{I}_g \to 0$.

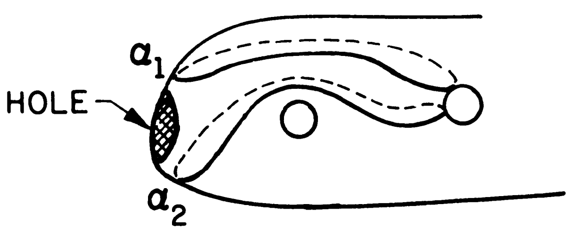

FIGURE 1

Now Powell's theorem says that $\mathcal{I}_g = \mathcal{W}_1 \cdot \mathcal{T}_1 \cdot \mathcal{T}_2$. But a $\mathcal{W}_1$ generator for a closed surface is $T_{\alpha_1} T_{\alpha_2}^{-1}$, where α_1, α_2 bound a surface of genus 1 on one

[1] **I am particularly indebted to Joan Birman for many interesting discussions concerning these matters and for encouraging this work.**

side and $g-2$ on the other. Suppose $D \subset M_g$ is a disc, and so its complement is $M_{g,1}$. We may change the α's by an isotopy so that they are disjoint from the disc D, and hence $T_{\alpha_1}T_{\alpha_2}{}^{-1}$ also defines a map in $\mathcal{I}_{g,1}$ lifting that in $\mathcal{I}_g$. Its genus is either 1 or $g-2$, depending on the position of the disc D with respect to α_1, α_2. Likewise, $\mathcal{T}_1$ generators of the closed surface may be lifted to $\mathcal{T}_1$ or $\mathcal{T}_{g-1}$ generators in $\mathcal{I}_{g,1}$, and $\mathcal{T}_2$ to $\mathcal{T}_2$ or $\mathcal{T}_{g-2}$. Since these lifted generators plus the generators of $\operatorname{Ker} p$ generate $\mathcal{I}_{g,1}$ we see that $\mathcal{I}_{g,1}$ is generated by the $\mathcal{W}_k$'s and $\mathcal{T}_k$'s. Our program will be to show that all of the $\mathcal{T}_k$'s are contained in $\mathcal{T}_1 \cdot \mathcal{T}_2$, and that $\mathcal{T}_1$, $\mathcal{T}_2$ and the $\mathcal{W}_k$'s are all contained in $\mathcal{W}_1$. This result will also clearly hold for $\mathcal{I}_g$ as well.

III. First we show that $\mathcal{W}_k \subset \mathcal{W}_1$. Consider Figure 2: the genus is $\geqslant 3$, and $g(\alpha_1, \alpha_2) = g(\alpha_2, \alpha_3) = 1$, $g(\alpha_1, \alpha_3) = 2$. Hence $T_{\alpha_1}T_{\alpha_2}{}^{-1}$ and $T_{\alpha_2}T_{\alpha_3}{}^{-1}$ are in $\mathcal{W}_1$, and their product $T_{\alpha_1}T_{\alpha_3}{}^{-1}$ is also; but the latter is a typical generator of $\mathcal{W}_2$. Since all these generators are conjugate in $\mathcal{M}$, and $\mathcal{W}_1$ is normal, we get $\mathcal{W}_2 \subset \mathcal{W}_1$. By induction (extending the genus of the surface between α_2 and α_3) we get $\mathcal{W}_k \subset \mathcal{W}_1$ for all possible k (that is, $k \leqslant g-1$).

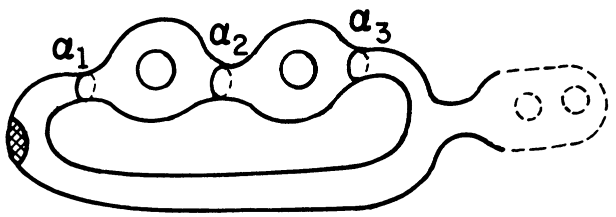

FIGURE 2

IV. We now introduce our main tool, a relation arising from twists on a sphere with four holes (that is, a disc with three holes): see Figure 3. Each ε_i is a curve parallel to a boundary curve and interior to the surface. The twists about the various curves can now be defined in the standard way. Here are our conventions. First, T_α means the homeomorphism which affects an arc crossing α by causing it to turn *right* as it approaches α, run once around α, and then progress on as before. Second, the order of composition we are using is the *functional* one: $T_\beta T_\alpha$ means apply T_α first, then T_β. Finally, recall that an equation between twist products means that the two sides are isotopic by an isotopy fixing the boundary. With these conventions then, the following relation holds:

$$T_\gamma T_\beta T_\alpha = T_{\varepsilon_1}T_{\varepsilon_2}T_{\varepsilon_3}T_{\varepsilon_4}.$$

We can prove the relation by looking at the effect of the map

$$T_\gamma T_\beta T_\alpha T_{\varepsilon_1}{}^{-1}T_{\varepsilon_2}{}^{-1}T_{\varepsilon_3}^{-1}T_{\varepsilon_4}{}^{-1}$$

on Figure 4 will find that the result can be deformed modulo the boundary back to the original. Since cutting Figure 4 along the three arcs reduces it to a disc, any map which fixes the boundary *and* arcs must be isotopic to 1; we omit the details.

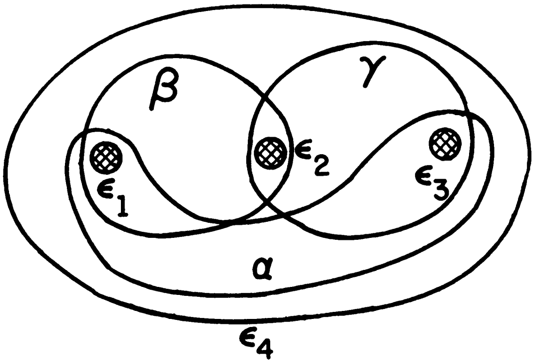

FIGURE 3

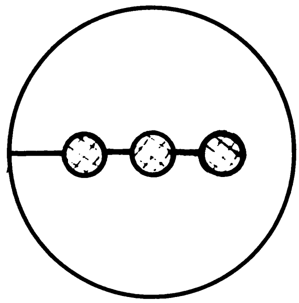

FIGURE 4

We now deform Figure 3 into Figure 5. We will use this lanternlike figure and its relation to derive relations in $\mathcal{G}$ by the process of glueing various surfaces (with one hole) onto the various holes ε_i of the lantern. Note that since each ε_i is disjoint from all other twist curves in the figure, it commutes with all of them. Hence in the relation

$$T_\gamma T_\beta T_\alpha T_{\varepsilon_1}{}^{-1} T_{\varepsilon_2}{}^{-1} T_{\varepsilon_3}{}^{-1} T_{\varepsilon_4}{}^{-1} = 1,$$

the T_{ε_i}'s may be placed anywhere and in any order.

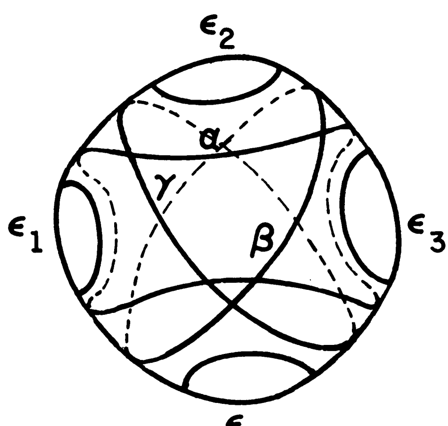

FIGURE 5

V. To begin with we glue surfaces of genus 1, 1, k to the curves ε_1, ε_2, ε_3 respectively of the lantern: see Figure 6. We have then:

$$g(\alpha) = g(\gamma) = k + 1; \qquad g(\beta) = 2,$$
$$g(\varepsilon_1) = g(\varepsilon_2) = 1, \qquad g(\varepsilon_3) = k, \qquad g(\varepsilon_4) = k + 2.$$

Here the lantern relation gives us:

$$T_{\varepsilon_4} = T_{\varepsilon_1}T_{\varepsilon_2}T_{\varepsilon_3}T_\gamma T_\beta T_\alpha \in \mathfrak{T}_1 \cdot \mathfrak{T}_2 \cdot \mathfrak{T}_k \cdot \mathfrak{T}_{k+1}$$

and T_{ε_4} is a generator of $\mathfrak{T}_{k+2}$; thus $\mathfrak{T}_{k+2} \subset \mathfrak{T}_1 \cdot \mathfrak{T}_2 \cdot \mathfrak{T}_k \cdot \mathfrak{T}_{k+1}$. By induction, we get $\mathfrak{T}_k \subset \mathfrak{T}_1 \cdot \mathfrak{T}_2$ for all $3 \leqslant k \leqslant g$, proving:

THEOREM 1. *If $\mathfrak{T}$ is the group generated by all twists on BSCC's then* $\mathfrak{T} = \mathfrak{T}_1 \cdot \mathfrak{T}_2$.

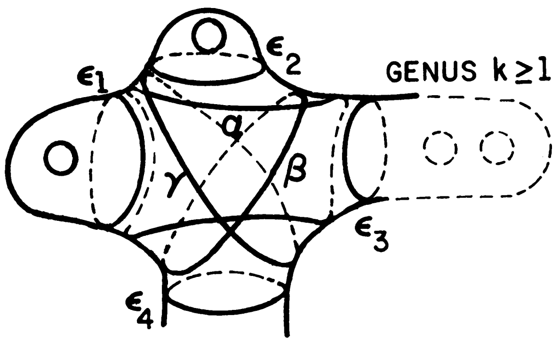

FIGURE 6

Suppose next we delete the genus 1 surface glued to ε_1 in Figure 6, and assume that the resulting figure sits in our surface M so that $\varepsilon_1 \not\simeq 0$ (this can be so arranged for $g \geqslant 3$). We still have $\alpha \simeq \varepsilon_4 \simeq \beta \simeq \varepsilon_1$, and get:

$$g(\alpha, \varepsilon_4) = g(\beta, \varepsilon_1) = 1, \qquad g(\varepsilon_2) = 1, \qquad g(\varepsilon_3) = k, \qquad g(\gamma) = k + 1.$$

Writing the lantern relation as

$$T_\gamma = T_{\varepsilon_2}T_{\varepsilon_3}\left(T_{\varepsilon_4}T_\alpha^{-1}\right)\left(T_{\varepsilon_1}T_\beta^{-1}\right) \in \mathfrak{T}_1 \cdot \mathfrak{T}_k \cdot \mathfrak{W}_1,$$

we see that $\mathfrak{T}_{k+1} \subset \mathfrak{W}_1 \cdot \mathfrak{T}_1 \cdot \mathfrak{T}_k$. In particular, for $k = 1$ we get $\mathfrak{T}_2 \subset \mathfrak{W}_1 \cdot \mathfrak{T}_1$. Finally, let us also remove the genus k surface from ε_3, and assume (again possible for $g \geqslant 3$) that the resulting figure, which now has only one surface of genus 1 glued on (to ε_2), is imbedded in M so that ε_1, ε_3 and ε_4 are all nonhomologous to zero. Then we get for the genera of homologous pairs:

$$g(\alpha, \varepsilon_4) = g(\beta, \varepsilon_1) = g(\gamma, \varepsilon_3) = 1, \qquad g(\varepsilon_2) = 1.$$

The lantern relation now reads:

$$T_{\varepsilon_2} = \left(T_\gamma T_{\varepsilon_3}{}^{-1}\right)\left(T_\beta T_{\varepsilon_1}{}^{-1}\right)\left(T_\alpha T_{\varepsilon_4}{}^{-1}\right) \in \mathfrak{W}_1,$$

and hence $\mathcal{T}_1 \subset \mathcal{WS}_1$. This shows us then that $\mathcal{T}_1 \cdot \mathcal{T}_2 \subset \mathcal{WS}_1$ and so $\mathcal{T}_k \subset \mathcal{WS}_1$ for all k. Using the results of §§II and III, we get finally:

THEOREM 2. $\mathcal{I} = \mathcal{WS}_1$.

Note that what we have actually proved is that the group generated by all of the $\mathcal{WS}_k$'s and $\mathcal{T}_k$'s is generated by $\mathcal{WS}_1$ alone. A direct proof that $\mathcal{I}$ is the former group would then give us Theorem 2 without the use of Powell's theorem, which is proved indirectly using a nongeometric argument concerning a presentation of the symplectic group $\mathrm{Sp}(g, Z)$.

VI. We are now in a position to show that $[\mathcal{M}, \mathcal{I}] = \mathcal{I}$. Consider Figure 7, let p be the 180° rotation around the central axis; we get $p(\alpha) = \alpha', p(\beta) = \beta'$. Hence $T_{\beta'} = pT_\beta p^{-1}$ and so $f = T_\beta T_{\beta'}^{-1} = [T_\beta, p] \in \mathcal{I}$ is a generator of $\mathcal{WS}_1$. Let h be any homeomorphism such that $h(\beta) = \alpha$, so that $T_\alpha = hT_\beta h^{-1}$. Then mod$[\mathcal{M}, \mathcal{I}]$ we get

$$[T_\beta, p] \equiv h[T_\beta, p]h^{-1} = [hT_\beta h^{-1}, hph^{-1}] = [T_\alpha, p[p^{-1}, h]].$$

Notice that the action of p on $H_1(M, Z)$ is just negation. Thus its action commutes with any linear map on H_1, so $[p^{-1}, h] = 1$ on H_1, i.e., $[p^{-1}, h] = k$ is in $\mathcal{I}$. Now using the standard commutator identity

$$[T_\alpha, pk] = [T_\alpha, p] \cdot p[T_\alpha, k]p^{-1}$$

and, noticing that $p[T_\alpha, k]p^{-1} \in [\mathcal{M}, \mathcal{I}]$, we get

$$f = [T_\beta, p] \equiv [T_\alpha, p] \bmod [\mathcal{M}, \mathcal{I}].$$

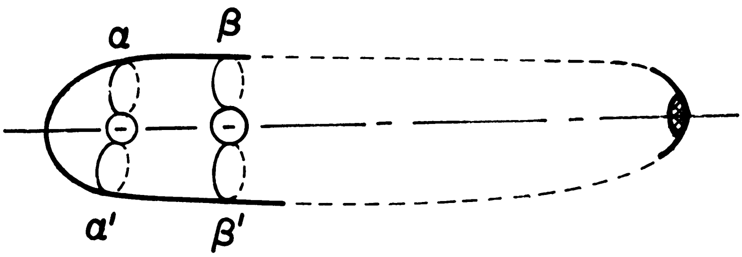

FIGURE 7

But $[T_\alpha, p] = T_\alpha T_{\alpha'}^{-1} = 1$ since α and α' are isotopic; thus $f \in [\mathcal{M}, \mathcal{I}]$. Since $[\mathcal{M}, \mathcal{I}]$ is normal in $\mathcal{M}$, this shows that $\mathcal{WS}_1 \subset [\mathcal{M}, \mathcal{I}]$, that is $\mathcal{I} \subset [\mathcal{M}, \mathcal{I}]$, that is:

THEOREM 3. $[\mathcal{M}, \mathcal{I}] = \mathcal{I}$.

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