PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 1, June 1979

HYPERSPACES HOMEOMORPHIC TO HILBERT SPACE

D. W. CURTIS[1]

ABSTRACT. The hyperspace 2^X of a metric space X is the space of nonempty compact subsets, topologized by the Hausdorff metric. It is shown that 2^X is homeomorphic to the separable Hilbert space l^2 if and only if X is connected, locally connected, separable, topologically complete, and nowhere locally compact. The principal tool in the proof is Torunczyk's mapping characterization of l^2.

1. Hyperspace characterization theorems. For a metric space (X, d), the hyperspace 2^X is the space of nonempty compact subsets of X, topologized by the Hausdorff metric $\tilde{d}(A, B) = \inf\{\varepsilon > 0: A \subset N(B; \varepsilon) \text{ and } B \subset N(A; \varepsilon)\}$. The topology induced by $\tilde{d}$ is independent of the particular metric d on X, since it is the same as the Vietoris finite topology on 2^X. In the finite topology the basic open sets are those of the form $V(G_1, \ldots, G_n) = \{F \in 2^X: F \cap G_i \neq \varnothing$ for each i and $F \subset \bigcup_1^n G_i\}$, where $G_1, \ldots, G_n$ are open sets in X.

In this paper we give necessary and sufficient conditions on X for the hyperspace 2^X to be homeomorphic to the separable Hilbert space l^2. This completes the following list of hyperspace characterization theorems for separable, topologically complete metric spaces.

THEOREM A (Wojdyslawski [9], Tasmetov [7]). *2^X is a separable, topologically complete metric AR if and only if X is connected, locally connected, separable, and topologically complete.*

THEOREM B (Curtis and Schori [3], [4]). *2^X is homeomorphic to the Hilbert cube $Q = \prod_1^\infty [-1, 1]$ if and only if X is nondegenerate, connected, locally connected, and compact.*

THEOREM C (Curtis [2]). *2^X is homeomorphic to $Q \setminus pt \approx Q \times [0, \infty)$ if and only if X is noncompact, connected, locally connected, and locally compact.*

THEOREM D ([2]). *X admits a Peano compactification $\hat{X}$ such that the hyperspace pair $(2^{\hat{X}}, 2^X)$ is homeomorphic to the pair $(Q, s) = (\prod_1^\infty[-1, 1], \prod_1^\infty(-1, 1))$ if and only if X is connected, locally connected, separable, topologically complete, nowhere locally compact and admits a metric with Property* S. (*A metric d on X has Property* S *if there exist finite covers of X by connected subsets with arbitrarily small diameters.*)

Received by the editors June 26, 1978 and, in revised form, August 28, 1978.
AMS (MOS) subject classifications (1970). Primary 54B20, 57A20; Secondary 54C55, 54D05.
[1]Supported by NSF grant MCS 76-06522.

THEOREM E. *2^X is homeomorphic to l^2 if and only if X is connected, locally connected, separable, topologically complete, and nowhere locally compact.*

Since the pseudo-interior $s = \prod_1^\infty(-1, 1)$ of the Hilbert cube is homeomorphic to l^2 (Anderson **[1]**), Theorem D gives sufficient (but not necessary) conditions on X for 2^X to be homeomorphic to l^2. The following examples illustrate the difference between (D) and (E).

EXAMPLE 1. Let $X = \bigcup_1^n s_i$, a finite union of copies of s, meeting at a single point. Then X does admit a metric with Property S, and the compactification $\hat{X} = \bigcup_1^n Q_i$, a union of n copies of Q meeting at a single point, has the property that $(2^{\hat{X}}, 2^X) \approx (Q, s)$. Thus in particular $2^X \approx s \approx l^2$. (This result, for $n = 1$, was first proved by Kroonenberg **[6]**.)

EXAMPLE 2. Let $X = \bigcup_1^\infty s_i$, a countably infinite union of copies of s meeting at a single point θ. We give X the uniform topology at θ (i.e., if f: $X \to s$ is a function such that the restrictions to each s_i are homeomorphisms onto s, then the basic neighborhoods of θ in X are of the form $f^{-1}(U)$, where U is a neighborhood of $f(\theta)$ in s). This space satisfies the hypotheses of (E), but it is easy to see that there is no metric with Property S. Thus, while X does have some Peano compactifications, there is no compactification $\hat{X}$ such that $(2^{\hat{X}}, 2^X) \approx (Q, s)$. It was shown by an ad hoc argument in **[2]** that $2^X \approx s \approx l^2$.

2. Torunczyk's characterization of Hilbert space. The proof of Theorem E is based on the mapping characterization of l^2 given recently by Torunczyk.

DEFINITION. If $\mathfrak{U}$ is an open cover of a space Y, then maps f,g: $X \to Y$ are said to be *$\mathfrak{U}$-close* if, for each x in X, there exists an element $\mathfrak{U}_x$ of $\mathfrak{U}$ containing both $f(x)$ and $g(x)$.

DEFINITION. A family $\{A_i\}$ of closed subsets in a space Y is *discrete* if each point of Y has a neighborhood meeting at most one member of $\{A_i\}$. (Equivalently, the family $\{A_i\}$ is pairwise disjoint and locally finite in Y.)

THEOREM (**[8]**). *Let Y be a separable, topologically complete metric AR. Then Y is homeomorphic to l^2 if and only if the following condition is satisfied*:

($*$) *For every map f: $\Sigma_1^\infty K_i \to Y$ of a countable disjoint sum of finite complexes, and every open cover $\mathfrak{U}$ of Y, there exists a map g: $\Sigma_1^\infty K_i \to Y$ such that f and g are $\mathfrak{U}$-close and $\{g(K_i)\}$ is discrete.*

3. Proof of Theorem E–Part I. Suppose $2^X \approx l^2$. Since l^2 is an AR, Theorem A implies that X is connected, locally connected, separable, and topologically complete. If some point p in X has a compact neighborhood, then $\{p\}$ has a compact neighborhood in 2^X. But this is impossible since l^2 is nowhere locally compact. Thus X must also be nowhere locally compact.

Conversely, suppose X satisfies the conditions of Theorem E. Then 2^X is a separable, topologically complete metric AR, and it remains to verify the condition ($*$) of Torunczyk's characterization theorem.

Let a map $f\colon \Sigma_1^\infty K_i \to 2^X$ and an open cover $\mathcal{U}$ of 2^X be given. For the reader's convenience, we first consider the special case in which each complex K_i is a point and $\mathcal{U}$ is the uniform cover by ε-neighborhoods. (The main idea of the more general proof is particularly transparent in this case.) Thus, f is simply a sequence $\{F_i\}$ in 2^X, and we must construct a discrete sequence $\{G_i\}$ such that $\tilde{d}(F_i, G_i) < \varepsilon$ for each i.

Let $\mathcal{V}$ be a locally finite (therefore countable) open cover of X with mesh $\mathcal{V} < \varepsilon$. Since X is nowhere locally compact, we may choose in each element of $\mathcal{V}$ an infinite discrete sequence. The union of these sequences over all the elements of $\mathcal{V}$ is a countable discrete subset Z of X. Each subset G_i will be of the form $G_i = F_i \cup \{z_i\}$, for some $z_i \in Z$. The points z_i are chosen inductively so that

(1) $d(z_i, F_i) < \varepsilon$,

(2) $z_i \notin \bigcup_{j<i} G_j$.

This is easily accomplished by considering any element V_i of $\mathcal{V}$ meeting F_i, then choosing $z_i \in V_i \cap Z \setminus \bigcup_{j<i} G_j$, which is possible since $V_i \cap Z$ is an infinite discrete set and $\bigcup_{j<i} G_j$ is compact. Then $\tilde{d}(F_i, G_i) = d(z_i, F_i) < \varepsilon$ for each i, and $G_i \neq G_j$ if $i \neq j$. If $\{G_i\}$ is not discrete, there exists a convergent subsequence $G_{i_n} \to G \in 2^X$. Then the sequence $\{z_{i_n}\}$ must cluster at some point in G. But $\{z_{i_n}\} \subset Z$ is discrete. Thus $\{G_i\}$ is a discrete sequence.

4. Proof of Theorem E–Part II. We now return to the general case in which each K_i is a finite complex and $\mathcal{U}$ is an arbitrary open cover of 2^X. A map $g\colon \Sigma_1^\infty K_i \to 2^X$ must be constructed such that f and g are $\mathcal{U}$-close and $\{g(K_i)\}$ is a discrete family in 2^X. This means that $g(K_i)$ and $g(K_j)$ have no common elements if $i \neq j$, and that every sequence $\{G_i\}$ in 2^X, where each $G_i \in g(K_i)$, is discrete.

For $A \in 2^X$ and $\mu > 0$, let $N(A; \mu) = \{x \in X\colon d(x, A) < \mu\}$, and $\tilde{N}(A; \mu) = \{F \in 2^X\colon \tilde{d}(A, F) < \mu\}$.

Step 1. There exists a map $\mu\colon 2^X \to (0, \infty)$ such that, for each $A \in 2^X$, $\tilde{N}(A; \mu(A))$ is contained in some element of $\mathcal{U}$. (Take $\mu(A) = \frac{1}{2}\sup\{\mu > 0\colon \tilde{N}(A; \mu) \subset U$ for some $U \in \mathcal{U}\}$.) Thus $f, g\colon \Sigma_1^\infty K_i \to 2^X$ will be $\mathcal{U}$-close provided $\tilde{d}(f(y), g(y)) < \mu f(y)$ for each y in $\Sigma_1^\infty K_i$.

Step 2. There exists a map $\eta\colon 2^X \to (0, \infty)$ such that if $x \in N(A; \eta(A))$, there exists an arc J in X connecting x and A, with $\operatorname{diam} J < \frac{1}{2}\mu(A)$. Necessarily, $\eta(A) \leqslant \frac{1}{2}\mu(A)$. (Use the local arc-connectedness of X and the compactness of elements of 2^X. For $A \in 2^X$ define $\bar{\eta}(A) = \sup\{\eta > 0\colon$ there exists $0 < \varepsilon < \frac{1}{2}\mu(A)$ such that if $x \in N(A; \eta)$, there exists an arc J in X connecting x and A, with $\operatorname{diam} J < \varepsilon\}$. Then $\bar{\eta}\colon 2^X \to (0, \infty)$ is lower semicontinuous, and there exists a continuous function η such that $0 < \eta(A) < \bar{\eta}(A)$ for each $A \in 2^X$ (see [**5**, p. 170]).

Step 3. Subdivide each complex K_i so that the given map $f\colon \Sigma_1^\infty K_i \to 2^X$

satisfies the following conditions for each simplex σ:

(i) $\operatorname{diam} f(\sigma) < \frac{1}{2} \inf_\sigma \eta f$, where $\operatorname{diam} f(\sigma) = \sup\{\tilde{d}(f(y), f(y')) : y, y' \in \sigma\}$, $\sigma\}$,

(ii) $\sup_\sigma \eta f < 2 \inf_\sigma \eta f$,

(iii) $\sup_\sigma \mu f < \frac{3}{2} \inf_\sigma \mu f$.

Step 4. For each n, let $\mathcal{V}_n$ be a locally finite (therefore countable) open cover of X with mesh $\mathcal{V}_n < 1/n$. In each element of $\mathcal{V}_n$ choose an infinite discrete subset of X. Then the union of such subsets over all the elements of $\mathcal{V}_n$ is an infinite discrete subset $Z(n)$ of X.

Step 5. We define g first at each vertex of the complex K_1. For a vertex v let n_v be the smallest integer such that $1/n_v < \frac{1}{4}\, \eta f(v)$, and choose a point z_v in $Z(n_v)$ such that $d(z_v, f(v)) = \min\{d(z_v, x) : x \in f(v)\} < 1/n_v$. Set $g(v) = f(v) \cup \{z_v\}$.

Step 6. We now extend g over the 1-skeleton $K_1^{(1)}$ of K_1. Let τ be an edge of K_1, with endpoints v and w and midpoint τ^*. Since

$$\begin{aligned} d(z_v, f(\tau^*)) &\leqslant d(z_v, f(v)) + \tilde{d}(f(v), f(\tau^*)) \leqslant \tfrac{1}{4}\, \eta f(v) + \tfrac{1}{2} \inf_\tau \eta f \\ &\leqslant \tfrac{1}{2} \inf_\tau \eta f + \tfrac{1}{2} \inf_\tau \eta f \leqslant \eta f(\tau^*), \end{aligned}$$

there exists an arc J_v (possibly degenerate) in X connecting z_v and $f(\tau^*)$, with $\operatorname{diam} J_v < \frac{1}{2} \mu f(\tau^*)$. Likewise, there exists an arc J_w connecting z_w and $f(\tau^*)$, with $\operatorname{diam} J_w < \frac{1}{2} \mu f(\tau^*)$. Define $g(\tau^*) = f(\tau^*) \cup J_v \cup J_w$. Then $\tilde{d}(f(\tau^*), g(\tau^*)) \leqslant \frac{1}{2} \mu f(\tau^*)$.

Let $h\colon I \to 2^X$ be a path from $f(\tau^*)$ to $g(\tau^*)$ such that $h(0) = f(\tau^*)$, $h(1) = g(\tau^*)$, and $f(\tau^*) \subset h(t) \subset g(\tau^*)$ for each t. Define g on the segment $[v\tau^*]$ (and similarly on the segment $[w\tau^*]$) as follows:

$$g((1 - t)v + t\tau^*) = \begin{cases} f((1 - 2t)v + 2t\tau^*) \cup \{z_v\} & \text{if } 0 \leqslant t \leqslant 1/2, \\ h(2t - 1) \cup \{z_v\} & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Thus g is defined over each edge $\tau = [v\tau^*] \cup [w\tau^*]$. We claim that for each $p \in \tau$, $\tilde{d}(f(\tau^*), g(p)) \leqslant \frac{1}{2}\, \mu f(\tau^*)$. With $p = (1 - t)v + t\tau^*$, suppose first that $0 \leqslant t \leqslant 1/2$. Then

$$\tilde{d}(f(\tau^*), f((1 - 2t)v + 2t\tau^*)) \leqslant \operatorname{diam} f(\tau) < \tfrac{1}{2} \inf_\tau \eta f \leqslant \tfrac{1}{2}\, \eta f(\tau^*),$$

and as shown above, $d(z_v, f(\tau^*)) \leqslant \eta f(\tau^*)$. It follows that $\tilde{d}(d(\tau^*), f((1 - 2t)v + 2t\tau^*) \cup \{z_v\}) \leqslant \eta f(\tau^*) \leqslant \frac{1}{2} \mu f(\tau^*)$. Now suppose $\frac{1}{2} \leqslant t \leqslant 1$. Then $f(\tau^*) \subset g(p) \subset g(\tau^*)$, hence $\tilde{d}(f(\tau^*), g(p)) \leqslant \tilde{d}(f(\tau^*), g(\tau^*)) \leqslant \frac{1}{2}\, \mu f(\tau^*)$.

Step 7. Extend g over all of K_1 as follows. Let $\mathcal{C}_1$ denote the hyperspace of subcontinua of the 1-skeleton $K_1^{(1)}$. There exists a map $r\colon K_1 \to \mathcal{C}_1$ such that $r(p) = \{p\}$ for each p in $K_1^{(1)}$, and such that if σ is the carrier of the point y, then $r(y) \subset \sigma^{(1)}$. (First define r over the 2-skeleton $K_1^{(2)}$ as follows. Consider each 2-simplex σ as the cone over its boundary $\sigma^{(1)}$, with cone point c. For each point $y = tc + (1 - t)p$ in σ, where $p \in \sigma^{(1)}$, take $r(y)$ to be a subcontinuum of $\sigma^{(1)}$ centered at p and with length proportional to t. If $t = 1$, then $r(y) = r(c) = \sigma^{(1)}$. Similarly, r is extended inductively over the higher-dimensional skeleta.) Now define $g(y) = \bigcup\{g(p): p \in r(y)\}$.

We claim that $\tilde{d}(f(y), g(y)) \leqslant \mu f(y)$ for each y in K_1. Let σ be the carrier of y, and set $F_y = \bigcup\{f(\tau^*): \tau$ is an edge of σ meeting $r(y)\}$. Then $\tilde{d}(f(y), g(y)) \leqslant \tilde{d}(f(y), F_y) + \tilde{d}(F_y, g(y)) \leqslant \frac{1}{2}\inf_\sigma \eta f + \frac{1}{2}\sup_\sigma \mu f \leqslant \frac{1}{4}\inf_\sigma \mu f + \frac{3}{4}\inf_\sigma \mu f \leqslant \mu f(y)$.

Thus $g: K_1 \to 2^X$ is $\mathfrak{U}$-close to the restriction $f: K_1 \to 2^X$, and for each y in K_1 with carrier σ, $g(y)$ contains the point z_v in $Z(n_v)$ for at least one vertex v of σ. Replacing $g(y)$ by $f(y) \cup g(y)$, we may assume also that $f(y) \subset g(y)$ for each y.

Step 8. The map g is defined similarly over each complex K_i, with the stipulation that the points z_v are chosen from $Z(n_v) \setminus \bigcup\{g(y): y \in K_1 \cup \cdots \cup K_{i-1}\}$. This insures that the images $g(K_1), g(K_2), \ldots$ in 2^X are disjoint.

We claim that $\{g(K_i)\}$ is a discrete family in 2^X. Suppose not. Then there exists a convergent sequence $g(y_{n_i}) \to G \in 2^X$, where $y_{n_i} \in K_{n_i}$ for each i. For convenience assume $n_i = i$; thus $g(y_i) \to G$. Each $g(y_i)$ contains a point z_i of $Z(n_{v_i})$, for some vertex v_i of the carrier σ_i of y_i. Since every finite union $\bigcup_{n=1}^k Z(n)$ is a discrete subset of X, and since every subsequence of $\{z_i\}$ must cluster at some point of G, we must have $n_{v_i} \to \infty$. Since n_{v_i} was specified to be the smallest integer for which $1/n_{v_i} < \frac{1}{4}\eta f(v_i)$, it follows that $\eta f(v_i) \to 0$. Since $f(y_i) \subset g(y_i)$ for each i, some subsequence $\{f(y_{i_j})\}$ must converge to a subset F of G. Thus $\eta f(y_{i_j}) \to \eta(F) > 0$. But then $\eta f(v_{i_j}) \geqslant \inf_{\sigma_{i_j}} \eta f > \frac{1}{2}\sup_{\sigma_{i_j}} \eta f \geqslant \frac{1}{2}\eta f(y_{i_j}) \to \frac{1}{2}\eta(F) > 0$, hence $\eta f(v_i) \not\to 0$, a contradiction.

References

1. R. D. Anderson, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519.

2. D. W. Curtis, *Hyperspaces of noncompact metric spaces*, Compositio Math. (to appear).

3. D. W. Curtis and R. M. Schori, 2^X *and* $C(X)$ *are homeomorphic to the Hilbert cube*, Bull. Amer. Math. Soc. **80** (1974), 927–931.

4. ______, *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math. **101** (1978), 19–38.

5. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

6. N. Kroonenberg, *Pseudo-interiors of hyperspaces*, Compositio Math. **32** (1976), 113–131.

7. U. Tasmetov, *On the connectedness of hyperspaces*, Soviet Math. Dokl. **15** (1974), 502–504.

8. H. Torunczyk, *Characterizing Hilbert space topology* (preprint).

9. M. Wojdyslawski, *Retractes absolus et hyperespaces des continus*, Fund. Math. **32** (1939), 184–192.

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803