ON NONTRIVIAL CHARACTERISTIC CLASSES OF CONTACT FOLIATIONS

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Abstract. In this article we give some nontrivial realizations of secondary characteristic classes of contact foliations.

1. Introduction. Given any differentiable pseudogroup \( \Gamma \), we can define the \( \Gamma \)-foliations. In the category of \( \Gamma \)-foliations one can construct the secondary characteristic classes for \( \Gamma \)-foliations. The first step in studying characteristic classes is to show that they are nontrivial. Kamber and Tondeur [2]-[4] have given many nontrivial realizations of secondary characteristic classes in various geometric contexts. In this article we give some nontrivial characteristic classes for contact foliations.

In §2 we recall a construction of the characteristic classes for contact foliations. In §3 we compute some of the nontrivial characteristic classes of contact foliations.

2. Contact structures and contact foliations. Let \( (z, x^1, \ldots, x^n, y^1, \ldots, y^n) \) be coordinates in \( (2n+1) \)-dimensional euclidean space, \( \mathbb{R}^{2n+1} \), and let \( \gamma \) be the 1-form on \( \mathbb{R}^{2n+1} \) defined by

\[
\gamma = dz + \sum_{i=1}^{n} y^i dx^i - x^i dy^i.
\]

\( \gamma \) is called a contact form on \( \mathbb{R}^{2n+1} \), and a diffeomorphism \( f: U \to U' \), where \( U \) and \( U' \) are open subsets of \( \mathbb{R}^{2n+1} \), is called a contact transformation if and only if \( f^* \gamma = \tau \gamma \), where \( \tau \) is a nonzero real valued function on \( U \). The collection \( \Gamma \) of all such contact transformations forms a pseudogroup. It is called the contact pseudogroup on \( \mathbb{R}^{2n+1} \).

Let \( \Gamma \) be the contact pseudogroup on \( \mathbb{R}^{2n+1} \). A \( \Gamma \)-foliation \( \mathcal{F} \) of codimension \( 2n+1 \) on a finite dimensional manifold \( M \) is a covering \( \{ U_i \} \) together with submersion \( f_i: U_i \to \mathbb{R}^{2n+1} \) that satisfy the condition: for every point \( x \in U_i \cap U_j \) there exists an element \( g_{ij} \in \Gamma \) such that \( f_i = g_{ij} \circ f_j \) in some neighborhood of \( x \). The collection \( \{ U_i, f_i \} \) is called the atlas of the foliation. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be \( \Gamma \)-foliations on the manifolds \( M \) and \( N \), respectively. A morphism \( \mathcal{F}_1 \to \mathcal{F}_2 \) is a smooth mapping \( f: M \to N \) that satisfies the...
following condition: Let \( x \in M \), and let \( g_1: U \rightarrow R^{2n+1} \) and \( g_2: V \rightarrow R^{2n+1} \) (\( U \subset M, \ V \subset N \)) be the mappings of neighborhoods of \( x \) and \( f(x) \) that belong to the atlases of the foliations \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively. Then there is a diffeomorphism \( \alpha \in \Gamma \) such that \( \alpha \circ g_1 = g_2 \circ f \) in some neighborhood of \( x \).

Let \( \Gamma \) be the contact pseudogroup on \( R^{2n+1} \). We define \( G^k \) to be the set of all \( k \)-jets of mappings in \( \Gamma \) with source and target at \( 0 \). The group \( G^1 \) is called the linear isotropy group of \( \Gamma \). Given a \( \Gamma \)-foliation \( \mathcal{F} \) on a manifold \( M \), we let \( T(\mathcal{F}) \) denote the subfibration of \( T(M) \) tangent to the leaf of \( \mathcal{F} \). We define \( P_{\mathcal{F}} \rightarrow M \) to be the principal \( G^1 \)-bundle associated with \( T(M)/T(\mathcal{F}) \).

F. Kamber and P. Tondeur [4] have shown that the principal \( G^1 \)-bundle \( P_{\mathcal{F}} \rightarrow M \) is a foliated principal bundle. For foliated principal bundles they have the following results.

**Theorem 2.1** [4]. Let \( G \) be a Lie group, \( \mathfrak{g} \) the Lie algebra of \( G \). Let \( \pi: P \rightarrow M \) be a foliated principal bundle, \( H \subset G \) a closed subgroup and \( P' \) an \( H \)-reduction of \( P \) given by a section \( s: M \rightarrow P/H \) of the induced map \( \pi: P/H \rightarrow M \):

(i) There is a well-defined multiplicative homomorphism

\[ \Delta_*: H^r(W(\mathfrak{g}, H)_q) \rightarrow H^*_{DR}(M) \]

where \( q \) is the codimension of the foliation on \( M \), \( W(\mathfrak{g}, H)_q \) the truncated relative Weil algebra. \( \Delta_* \) is the generalized characteristic homomorphism of \( P \).

(ii) \( \Delta_* \) does not depend on the choice of an adapted connection in \( P \). But if \( P \) admits a basic connection, then

\[ \Delta_*: H^r(W(\mathfrak{g}, H)_{(q/2)}) \rightarrow H^*_{DR}(M). \]

(iii) \( \Delta_* \) is functorial under pullback and functorial in \((G, H)\).

(iv) \( \Delta_* \) is invariant under integrable homotopies.

3. Examples of nontrivial characteristic classes for contact foliations. Throughout this paper we shall use the notations, terminologies, and some results from [2], [4], [5], [7], without always specifying where they come from.

Let us fix an integer \( r \) satisfying \( 0 < r < [(n-1)/2] \). Let \( P^n(C) \) be the \( n \)-dimensional complex projective space, and let \( z_0, z_1, \ldots, z_n \) be the system of its homogeneous coordinates. We define the hermitian matrices \( I_r \) and \( \bar{I}_r \) of degree \( n - 1 \) and \( n + 1 \) by

\[ I_r = \begin{pmatrix} -E_r & 0 \\ 0 & E_{n-r-1} \end{pmatrix} \quad \text{and} \quad \bar{I}_r = \begin{pmatrix} 0 & 0 & \sqrt{-1} \\ 0 & I_r & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix} \]

where \( E_s \) is the unit matrix of degree \( s \).
Let $Q_r$ be the quadric of $P^n(C)$ defined by $\tilde{I}_r$, that is

$$Q_r = \left\{ (z_0, \ldots, z_n) \in P^n(C) \left| -\sqrt{-1} z_0 \bar{z}_n - \sum_{i=1}^{r} z_i \bar{z}_i + \sum_{i=r+1}^{n-1} z_i \bar{z}_i + \sqrt{-1} z_n \bar{z}_0 = 0 \right. \right\}.$$ 

It is known from [7] that $Q_r$ is a connected nondegenerate hypersurface of $P^n(C)$ and its index is $r$. We know that $Q_r$ has a contact structure of dimension $2n - 1$.

Let $P(n, C)$ be the group of all projective transformations. We consider the subgroup $G(r)$ of $P(n, C)$ which consists of all projective transformations leaving $Q_r$ invariant. $G(r)$ acts effectively and transitively on $Q_r$, and $G(r) = \text{SU}(\tilde{I}_r)/n$, where $n$ is the center of $\text{SU}(\tilde{I}_r)$. $G(r)$ is a connected Lie group. We denote by $G'(r)$ the isotropy subgroup of $G(r)$ at $0 = (1, 0, \ldots, 0) \in Q_r$.

The Lie algebra $g(r)$ of $G(r)$ can be identified with $\mathfrak{su}(\tilde{I}_r)$, that is,

$$g(r) = \left\{ X \in \mathfrak{gl}(n + 1, C) | [\bar{X} \bar{I}_r, + \bar{I}_r X] = 0, \text{trace} X = 0 \right\};$$

$g(r)$ is isomorphic to $\mathfrak{su}(r + 1, n - r)$, and so it is simple. Each element $X$ of $g(r)$ can be written as a matrix of the form

$$\begin{pmatrix}
-\bar{u} & -\sqrt{-1} \bar{w} & w_n \\
\xi & v & w \\
\xi_m & \sqrt{-1} \xi_r & u
\end{pmatrix}$$

where $\xi, w_n \in R, u \in C, \xi, w \in C^{n-1}, v \in \text{u}(I_r)$, and $u - \bar{u} + \text{trace} v = 0$.

The Lie algebra $g(r)$ has a graded Lie algebra structure $g(r) = \Sigma_k \mathfrak{g}_k(r)$, where the $\mathfrak{g}_k(r)$, $k = -2, -1, 0, 1, 2$ are defined as follows: Each element $X$ of $\mathfrak{g}_{-1}(r)$ can be represented as a matrix of the following form

$$X = \begin{pmatrix}
0 & 0 & 0 \\
\xi & 0 & 0 \\
0 & \sqrt{-1} \xi & 0
\end{pmatrix}$$

where $\xi \in C^{n-1}$. Each element $X$ of $\mathfrak{g}_1(r)$ can be represented as a matrix of the form

$$X = \begin{pmatrix}
0 & -\sqrt{-1} \xi & 0 \\
0 & 0 & \xi \\
0 & 0 & 0
\end{pmatrix}$$

where $\xi \in C^{n-1}$. Each element $X$ of $\mathfrak{g}_{-2}(r)$ can be represented as a matrix of
the form
\[ X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ a & 0 & 0 \end{bmatrix} \]
where \( a \in R \). Each element \( X \) of \( g_{2}(r) \) can be represented as a matrix of the form
\[ X = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
where \( a \in R \). Each element \( X \) of \( g_{6}(r) \) can be represented as a matrix of the form
\[ X = \begin{bmatrix} -\bar{u} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{bmatrix} \]
where \( u \in C, v \in u(I_{r}) \), and \( u - \bar{u} + \text{trace } v = 0 \).

Let \( \sigma \) be the involutive automorphism of \( g(r) \) which is defined by \( \sigma(X) = -X \) for \( X \in g(r) \). Then we have \( \sigma G(r) \sigma = G(r) \). We define a subgroup \( K(r) \) of \( G(r) \) by
\[ K(r) = \{ X \in G(r) | \sigma X \sigma = X \} \]
and put \( K_{0}(r) = K(r) \cap G'(r) \). Note that \( K(r) \) is compact and \( G(r)/G'(r) = K(r)/K_{0}(r) = \mathbb{Q}r \). The Lie algebra of \( K(r) \) is given by
\[ \mathfrak{k}(r) = \{ X \in g(r) | \sigma X = X \} \]
and \( \mathfrak{k}(r) = m(r) + \mathfrak{k}_{0}(r) \) (direct sum) where \( \mathfrak{k}_{0}(r) = \mathfrak{k}(r) \cap g_{0}(r) \) is the Lie algebra of \( K_{0}(r) \) and \( m(r) = (\sigma + 1) \mathfrak{p}(r) \) where \( \mathfrak{p}(r) = g_{-2}(r) + g_{-1}(r) \). We give some detailed description of the Lie algebra \( \mathfrak{k}(r) = \mathfrak{k}_{0}(r) + m(r) \). The Lie algebra \( \mathfrak{k}(r) \) is isomorphic to \( su(r + 1, n - r) \cap u(n + 1) \) and the Lie algebra \( \mathfrak{k}_{0}(r) \) is isomorphic to \( u(r) \times u(n - r - 1) \). We can represent each element \( X \) of \( \mathfrak{k}(r) \) as a matrix of the following form
\[ X = \begin{bmatrix} b\sqrt{-1} & -i\xi & -a \\ \xi & v & \sqrt{-1}I_{r}\xi \\ a & \sqrt{-1}i\xi I_{r} & b\sqrt{-1} \end{bmatrix} \]
where \( a, b \in R, \xi \in C^{n-1}, v \in u(r) \times u(n - r - 1) \), and \( 2b\sqrt{-1} + \text{trace } v = 0 \). Each element \( X \) of \( \mathfrak{k}_{0}(r) \) can be represented as a matrix of the following form
\[ X = \begin{bmatrix} b\sqrt{-1} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & b\sqrt{-1} \end{bmatrix} \]
where $b \in R$, $v \in u(r) \times u(n - r - 1)$, and $2b\sqrt{-1} + \text{trace } v = 0$. Each element $X$ of $m(r)$ can be represented as a matrix of the following form

$$X = \begin{bmatrix}
0 & -i\xi & -a \\
\xi & 0 & \sqrt{-1} I_{r}\xi \\
-\bar{a} & \sqrt{-1} i\xi & 0
\end{bmatrix}$$

where $a \in R$, $\xi \in C^{n-1}$.

Consider the Lie group $K(r)$ with the Lie subgroup $K_0(r) \subset K(r)$ and corresponding Lie algebra $\mathfrak{k}_0(r) \subset \mathfrak{k}(r)$. From [2], we have the following results. On the $K_0(r)$-bundle

$$P = K(r) \times K_0(r) \to K(r) \quad (3.1)$$

the $K_0(r)$-orbits of the diagonal $K_0(r)$-action on $K(r) \times K_0(r)$ defined by $(k, k)k' = (kk', k'^{-1}k)$ for $k \in K$, $k, k' \in K_0$ lift the $K_0(r)$-orbits on $K(r)$ (the left cosets) to $K(r) \times K_0(r)$. Thus the bundle (3.1) has a canonical foliated bundle structure. Moreover, the normal bundle $Q_{K_0(r)}$ of the homogeneous foliation $L_{K_0(r)}$ on $K(r)$ given by $K_0(r)$ (see [4]) is associated to $P$, and the canonical foliation on $Q_{K_0(r)}$ is inherited from the canonical foliation of $P$. Thus from here we see that $K(r)$ has a contact foliation structure of codimension $2n - 1$.

Now we consider the exact sequence

$$0 \to \mathfrak{k}_0(r) \to \mathfrak{k}(r) \to \mathfrak{k}(r)/\mathfrak{k}_0(r) \to 0.$$

Since $\mathfrak{k}(r) = \mathfrak{k}_0(r) + m(r)$, we define a map $\theta$ as the projection: $\mathfrak{k}(r) \to \mathfrak{k}_0(r)$. It is easy to verify that $\theta$ is a $K_0(r)$-equivariant splitting of the exact sequence. From [2] we know that $\theta$ defines a $L_{K_0(r)}$-basic connection of the $K_0(r)$-principal bundle (3.1).

In order to compute the map $\Delta_\ast$ in Theorem 2.1, we will need the following facts from [2], [4].

(I) Let $G \subset \tilde{G}$ be Lie groups. The canonical $G$-foliation $L_G$ of $M = \tilde{G}$ has codimension $q = \dim \tilde{G}/G$. The $G$-bundle $P = \tilde{G} \times G \to M$ is canonically foliated. Assume that $\theta$ is a $G$-equivariant splitting of the exact sequence

$$0 \to \mathfrak{g} \overset{\theta}{\to} \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}/\mathfrak{g} \to 0. \quad (3.2)$$

Then there exist an $L_G$-basic and locally $\tilde{G}$-invariant adapted connection $w$ on $P$, and $\Delta(w)$ on the cochain level factorized as follows

$$W(\mathfrak{g})_{\{q/2\}} \xrightarrow{\Delta(\theta)} \Lambda\tilde{\mathfrak{g}}^* \xrightarrow{\Gamma(M, \Omega^1)} \Lambda\tilde{\mathfrak{g}}^* \xrightarrow{\Delta(w)} \Gamma(M, \Omega^1).$$

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where \( \gamma \) is the canonical inclusion and \( \Delta(\theta) \) is induced by the \( G\)-DG-homomorphism

\[
\Delta(\theta) : W(\mathfrak{g}) \to \Lambda \tilde{\mathfrak{g}}^*
\]

which is completely determined by

\[
\Delta(\theta) \alpha = \alpha \theta \quad \text{for} \ \alpha \in \Lambda^1 \mathfrak{g}^*, \quad \Delta(Q) \alpha = \alpha K(\theta) = d_\lambda \alpha \theta + \frac{1}{2} \alpha[\theta, \theta] \quad \text{for} \ \tilde{\alpha} \in S^1 \mathfrak{g}^*.
\]

(II) Let \( (\tilde{G}, G) \) be a reductive pair and \( \theta : \tilde{g} \to \mathfrak{g} \) a \( G \)-equivariant splitting of (3.2). There is a commutative diagram of filtration preserving \( G\)-DG-homomorphism

\[
\begin{array}{ccc}
W(\mathfrak{g})_{q/2l} & \overset{\Delta(\theta)}{\to} & \Lambda \tilde{\mathfrak{g}}^* \\
\varphi \uparrow \cong & & \varphi \uparrow \cong \\
\Lambda P_\mathfrak{g} \otimes I(G)_{q/2l} & \to & \Lambda P_\mathfrak{g} \otimes (\Lambda m^*)^G
\end{array}
\]

where \( m = \ker \theta \), \( h(\theta) = \Delta(\theta)_G : I(G)_{q/2l} \to (\Lambda m^*)^G \) is induced by the (ordinary) characteristic homomorphism of \( (\tilde{G}, G) \) with values in the invariant forms \( (\Lambda m^*)^G \), and where the vertical maps are cohomology isomorphisms.

Now let \( (\tilde{G}, G) = (K(r), K_0(r)) \). Since \( K(r) \) is compact and connected, it is well known that \( \gamma^* : H^*(t(x)) \to H^*_DR(K(r)) \) is an isomorphism. By (I) the crucial map to evaluate is the cohomology map \( \Delta(\theta)_* \). If we identify \( t_0(r) \) with \( u(r) \times u(n - r - 1) \) by the matrices

\[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & v & 0 \\
0 & 0 & \lambda
\end{pmatrix}
\]

with \( v \in u(r) \times u(n - r - 1) \), \( \lambda = -\frac{1}{2} \) trace \( v \), then

\[
I(K_0(r)) \cong R[c_1, c_2, \ldots, c_{n-1}] \quad \text{and} \quad I(K(r)) \cong R[\tilde{c}_2, \ldots, \tilde{c}_{n+1}]
\]

where the \( c_i, \ i = 1, 2, \ldots, n - 1 \) (resp. \( \tilde{c}_j, \ j = 2, \ldots, n + 1 \)) are Chern polynomials. The restriction map \( i^* : I(K(r)) \to I(K_0(r)) \) is given by

\[
\begin{align*}
& I(K(r)) \quad \overset{i^*}{\to} \quad I(K_0(r)) \\
& R[\tilde{c}_2, \ldots, \tilde{c}_{n+1}] \quad \overset{i^*}{\to} \quad R[c_1, \ldots, c_{n-1}].
\end{align*}
\]

\[
i^*\tilde{c}_{i+1} = c_{i+1} - c_1 c_i + \frac{1}{4} c_1^2 c_{i-1}, \quad \text{for} \ \ i = 1, 2, \ldots, n - 2, \quad i^*\tilde{c}_{n} = \frac{1}{4} c_1^2 c_{n-2} - c_1 c_{n-1} \quad \text{and} \quad i^*\tilde{c}_{n+1} = \frac{1}{4} c_1^2 c_{n-1}.
\]

Let \( y_i = \sigma c_i \) be the suspension of \( c_i, \ i = 1, 2, \ldots, n - 1 \). Consider the element \( \tilde{c}_{n+1} \), a transgressive cochain \( Ti^*\tilde{c}_{n+1} \) of \( i^*\tilde{c}_{n+1} = \frac{1}{4} c_1^2 c_{n-1} \) can be obtained in the following way. The cochain

\[
z_{n+1}' = y_{n-1} \otimes \frac{1}{4} c_1^2 \in \Lambda(y_1, \ldots, y_n) \otimes R[c_1, \ldots, c_{n-1}]
\]
clearly satisfies
\[ dz'_{n+1} = 1 \otimes \frac{1}{4} c_1^2 c_{n-1} = 1 \otimes i^* \tilde{c}_{n+1} \]
and hence \( \varphi(z'_{n+1}) \in W(\mathfrak{g}(r)) \) is a transgressive cochain \( Ti^* \tilde{c}_{n+1} \) for \( i^* \tilde{c}_{n+1} \). But then by Proposition 5.22 of [2] and by naturality of \( \varphi \) we obtain the suspension of \( \tilde{c}_{n+1} \)

\[ \tilde{\sigma}_{n+1} = \left[ \Delta(\theta) Ti^* \tilde{c}_{n+1} - (\lambda^1 \tilde{c}_{n+1})^{2n+1,0} \right] = \left[ \Delta(\theta) Ti^* \tilde{c}_{n+1} \right] \]
\[ = \left[ \Delta(\theta) \varphi(z'_{n+1}) \right] = \left[ \varphi(z_{n+1}) \right] \in H^*(\tilde{\mathfrak{g}}) \] (3.3)

where
\[ \tilde{\sigma}_G(\lambda^1 \tilde{c}_{n+1}) = (\lambda^1 \tilde{c}_{n+1})^{2n+1,0} \in (\Lambda^{2n+1} \mathfrak{m}(r))^G = 0 \]
is the component of bidegree \((2n + 1, 0)\) of
\[ \lambda^1 \tilde{c}_{n+1} \in W(\tilde{\mathfrak{g}}, G) = (\Lambda \mathfrak{m}(r)^* \otimes S\tilde{\mathfrak{g}}^*)^G, \]
and
\[ z_{n+1} = y_{n-1} \otimes \frac{1}{4} h(c_1^2) \in \Lambda(y_1, \ldots, y_{n-1}) \otimes (\Lambda \mathfrak{m}(r)^*). \]

Since \((\Lambda^{2n+2} \mathfrak{m}(r)^*)^G\), the cochain \( z_{n+1} \) is a cocycle representing the suspension \( \tilde{\gamma}_{n+1} \) of \( \tilde{c}_{n+1} \).

The cochain \( z'_{n+1} = y_{n-1} \otimes \frac{1}{4} c_1^2 \) is a cocycle in \( \Lambda(y_1, \ldots, y_{n-1}) \otimes R[c_1, \ldots, c_{n-1}]_{n-1} \). From (3.3) the cocycle \( z'_{n+1} \) maps under \( \Delta(\theta) \sim \text{id} \otimes h \) into the cocycle \( z_{n+1} \) representing the nontrivial cohomology class \( \tilde{\gamma}_{n+1} \in H^*(\mathfrak{f}(r)) \approx \Lambda(\tilde{\mathfrak{g}}_2, \ldots, \tilde{\mathfrak{g}}_{n+1}) \).

Hence we have the following theorem.

**Theorem 3.1.** Let \((K(r), K_0(r))\) be the reductive pair which is defined as above. Then
\[ \tilde{\gamma}_{n+1} \in \text{Im } \Delta(\theta)_* \subset H^*(\mathfrak{f}(r)) \approx \Lambda(\tilde{\mathfrak{g}}_2, \ldots, \tilde{\mathfrak{g}}_{n+1}), \]
and \( \gamma_a(\tilde{\gamma}_{n+1}) \in \text{Im } \Delta_\ast \) is a nonzero characteristic class of the contact foliation of \( K(r) \) which is induced by the pair \((K(r), K_0(r))\).

**References**


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