THE RIGIDITY PROBLEM FOR STABLE SPACES

DONALD W. KAHN

Abstract. This note treats the problem of when the group of homotopy self-equivalences of a space is trivial. For stable spaces, with finitely many nonvanishing homotopy groups, we give a complete solution in an inductive sense. One of the consequences of this result is that for any stable space, with precisely two nonvanishing homotopy groups, the group of self-equivalences is nontrivial.

If $X$ is a (pointed) path connected space, we denote by $G(X)$ the group of homotopy classes of homotopy equivalences from $X$ to itself (all preserving base point). This is the homotopy theorist’s analogue of the group of automorphisms of a group, and it has now been studied by many authors. For example, C. Wilkerson [8] and D. Sullivan [7] have shown that when $X$ is a simply-connected finite complex, $G(X)$ is a finitely-presented group. Their methods, refining methods in [4], use algebraic groups, and they are not fine enough to get closer to $G(X)$ than a subgroup of finite index, or a quotient by a finite, normal subgroup. On the other hand, there is a space $X$, which at every prime but 3 is a $K(Z, 4)$, but for which $G(X)$ is trivial [5]. Since $G(K(Z, 4)) = Z_2$, the integers mod 2, we see that a space $X$ may be equivalent to a space $Y$, at all but one prime, yet $G(X)$ is trivial and $G(Y)$ is not. We shall call a space $X$, for which $G(X)$ is trivial, rigid, and we propose to study such spaces—in a stable sense—in this paper.

We shall work in a category whose objects are spaces with finitely-many nonvanishing homotopy groups, in a stable range. Specifically, such a space $X$ shall be endowed with a finite Postnikov tower

$$X = X_k \xrightarrow{\pi_k} \cdots \xrightarrow{\pi_{n+1}} X_n$$

where $\pi_i$ is a principal fibration with fibre $K(\pi_i(X), i)$, $n > 1$, and $k < 2n - 1$. Such a space is infinitely de-loopable, and our continuous maps and homotopies shall be presumed to be infinitely de-loopable also. Base points are assumed but omitted from the notation. We denote this category by $S_1$.

For comparison purposes—at the end of this note—we denote by $S_2$ the category of simply-connected finite complexes, and stable (in the sense of suspension) homotopy classes of maps.

In a given Postnikov system, if $j > i$, there is a natural homomorphism (see [2])

$$\phi_{j,i} : G(X_j) \to G(X_i),$$

Received by the editors June 29, 1978 and, in revised form, September 5, 1978.


Key words and phrases. Homotopy equivalences, stable spaces, Postnikov tower.
which is studied in [3]. The image of \( \phi_{m+1_m} \) is precisely known, while the kernel is there related to certain cohomology groups and extensions.

Our idea is now to study the rigidity problem, that is when a space is rigid, by induction. We easily characterize when \( K(\pi, n), \ n > 1 \), is rigid (Proposition 2), and then go on to give necessary and sufficient conditions for \( \ker(\phi_{m+1_m}) \) to be trivial. As a corollary, we may conclude that there is no space \( X \) in \( S_1 \), with precisely two nonvanishing homotopy groups, which is rigid. In conclusion, we compare this situation with that in the category \( S_2 \). We recall (Proposition 3) the well-known result that there is no nontrivial \( X \) in \( S_2 \), which is rigid. We then show (Proposition 4) that if \( X_k \) is a Postnikov term for a finite stable complex \( X \), and \( X_k \) is rigid, then \( \dim(X) \geq k \).

Finally, we would like to point out that our results for \( S_1 \) do not require the homotopy groups of the spaces in question to be finitely-generated, as opposed to many of the other papers in the field. For example, our Proposition 2 actually holds for all Abelian groups \( \pi \), if we choose \( X \) to be cellular.

**Proposition 1.** If \( X \) is a rigid space, in the category \( S_1 \), then the identity map \( 1_X \) has order 2 in the sense of loop multiplication in \( X \). In particular, every element in the homotopy or stable homology groups of \( X \) has order 2.

**Proof.** Let \([X, X]\) denote the Abelian group of homotopy classes of maps from \( X \) to itself, under loop multiplication. Assume that the order of the class of \( 1_X \) is different from 2. Then \( -1_X \) is not homotopic to \( 1_X \). As \( -1_X \) is clearly a homotopy equivalence, \( G(X) \) has at least two elements.

For the rest of the proposition, we simply observe that the addition in \([X, X]\) acts additively on homotopy and stable homology (that is homology groups in the stable range).

**Proposition 2.** Let \( X \) be a space in \( S_1 \), with a single, nonvanishing homotopy group, i.e., \( X = K(\pi, n), \ n > 1 \). Then \( X \) is rigid precisely when \( \pi = \mathbb{Z}_2 \), the integers mod 2.

**Proof.** If \( X \) is rigid, then we know that every element of \( \pi \) has order 2. I claim that \( \pi \) is a \( \mathbb{Z}_2 \)-vector space. It is trivial to construct a group homomorphism from a \( \mathbb{Z}_2 \)-vector space \( V \) onto \( \pi \), say \( g: V \to \pi \). But the kernel of \( g \) is trivially checked to be a vector subspace, proving that \( \pi \) is the quotient of a vector space by a subspace, and hence, \( \pi \) is itself a vector space.

Now, it is well known that \( G(K(\pi, n)) = \text{Aut}(\pi) \), the group of automorphisms of \( \pi \). But the only \( \mathbb{Z}_2 \)-vector space, without a nontrivial automorphism, has dimension 1.

Before stating our main theorem, it will be convenient to specify some notation.

a. \( \Sigma X \) means the (reduced) suspension of \( X \). \( \Sigma f \) means the suspension of the map \( f: \sigma: H^p(Y) \cong H^{p+1}(\Sigma Y) \) is the usual suspension isomorphism in cohomology. \( \Omega X \) means loops on \( X \).
b. If $Y$ is $(n-1)$-connected, we shall freely write $i_n \in H^n(Y; \pi_n(Y))$ for the fundamental class. If $x \in H^m(Y; \pi)$, write $\phi_x : Y \to K(\pi, m)$ for a map such that $\phi_x^*(i_m) = x$.

c. For a fibration in a given Postnikov tower, say

$$X_{n+1} \xrightarrow{\pi_{n+1}} X_n,$$

we shall abbreviate the fibre $K(\pi_{n+1}(X), n + 1)$ to $K$. We write the inclusion of the fibre $i : K \to X_{n+1}$.

**Theorem.** Let $X$ be a space in $\mathcal{S}_1$, and let

$$\phi_{m+1,m} : G(X_{m+1}) \to G(X_m)$$

be the homomorphism described above. Then a necessary and sufficient condition that $\ker(\phi_{m+1,m})$ be trivial, is that for every cohomology class

$$x \in H^{m+1}(X_{m+1}; \pi_{m+1}(X)),$$

for which $i \cdot \phi_x + 1_{X_{m+1}}$ induces an isomorphism on homology in dimension $m + 1$, there is a map

$$\alpha : \Sigma X_{m+1} \to X_m$$

so that $\alpha^*(k^{m+2}) = \sigma(x)$, with $k^{m+2}$ being the $k$-invariant of the fibration.

**Proof.** We are working in the stable category $\mathcal{S}_1$. If we suppose $\{f\} \in \ker(\phi_{m+1,m})$, then the diagram

$$\begin{array}{ccc}
X_{m+1} & \xrightarrow{f-1} & X_{m+1} \\
\downarrow \pi_{m+1} & & \downarrow \pi_{m+1} \\
X_m & \xleftarrow{0} & X_m
\end{array}$$

is homotopy commutative ($0$ sends all $X_{m+1}$ to the base point).

By exactness, there is a map $\psi_f : X_{m+1} \to K = \pi_{m+1}^{-1}(pt.)$, so that

$$\begin{array}{ccc}
K & \xrightarrow{\alpha} & X_m \\
\downarrow \psi_f & & \downarrow \psi_f \\
X_{m+1} & \xleftarrow{f-1} & X_{m+1}
\end{array}$$

is homotopy commutative.

Now, the fibration $\pi_{m+1}$ gives rise to an exact sequence of spaces

$$\cdots \to \Omega X_m \to K \xrightarrow{i} X_{m+1} \xrightarrow{\pi_{m+1}} X_m \to \cdots,$$

it is well known that if we identify cohomology of $\Sigma \Omega X_m$ and $X_m$ in the stable range, then $\sigma(\gamma^*(i_{m+1})) = k^{m+2}$, the $k$-invariant.

By exactness, the map $f - 1_{X_{m+1}} \simeq i \cdot \psi_f$ will be null-homotopic (and hence $f \simeq 1_{X_{m+1}}$) precisely when there is a map $\alpha' : X_{m+1} \to \Omega X_m$, with
being homotopy commutative.

If we suspend this diagram and write \( \alpha = \Sigma \alpha' \), then stably we get a homotopy commutative diagram

\[
\begin{array}{ccc}
X_{m+1} & \xrightarrow{\alpha'} & \Omega X_m \\
\downarrow{\psi_f} & & \downarrow{\gamma} \\
K & \xrightarrow{} & K (\pi_{m+1} (X), m + 2)
\end{array}
\]

On cohomology, \( \alpha^* (k^{m+2}) = (\Sigma \psi_f)^* (i_{m+2}) = \sigma (\psi_f^* (i_{m+1})) \).

To complete the proof, we must see that the classes \( x \), with the condition given in the statement of the theorem, are precisely those classes of the form \( \psi_f^* (i_{m+1}) \). But given such a class \( x \), \( i \cdot \phi_x + 1_{X_{m+1}} \) clearly induces isomorphisms on homology in dimensions through \( m + 2 \) (recall \( H_{m+2} (K) = 0 \)) and is thus a homotopy equivalence \( f \). Then \( f - 1_{X_{m+1}} \simeq i \cdot \phi_x \), so we take \( \phi_x = \psi_f \) and \( \psi_f^* (i_{m+1}) = \phi_x^* (i_{m+1}) = x \). On the other hand, given \( \psi_f \) for a homotopy equivalence \( f \), with \( \{ f \} \in \ker (\phi_{m+1,m}) \)–the class \( \psi_f^* (i_{m+1}) \) clearly meets the condition of the theorem.

We remark that the theorem is clearly valid for \( k (\phi_{j,i}) \), \( j > i \), when the homotopy groups in dimensions between \( i \) and \( j \) vanish.

**Corollary.** If \( X \) is a space in \( \mathbb{S}_1 \) with precisely two nonvanishing homotopy groups, \( X \) is not rigid.

**Proof.** By our propositions, we assume the two nonvanishing groups \( \pi_i (x) \) and \( \pi_j (x) \), \( i < j < 2i - 1 \), are both direct sums of copies of \( \mathbb{Z}_2 \). We shall prove \( \ker (\phi_{j,i}) \) is nontrivial.

By [6], the \( \mathbb{Z}_2 \)-cohomology of \( K (\mathbb{Z}_2, i) \) is nonzero in every dimension not less than \( i \). In particular, the subgroup

\[
H^j \left( K (\pi_i (X), i); \pi_j (X) \right) \subseteq H^j \left( X_j; \pi_j (X) \right)
\]

is nontrivial. Let \( x \neq 0 \) be a class in this subgroup. Consider the map

\[
X_j \xrightarrow{\phi_x \times 1_x} K (\pi_j (X), j) \times X \xrightarrow{\mu} X,
\]

where \( \mu \) is the action of the fibre. It clearly induces isomorphisms on homotopy groups and is thus a homotopy equivalence. Stably, it is just \( i \cdot \phi_x + 1_X \).

We must ask whether there is \( \alpha : \Sigma X \to X_i \) with \( \alpha^* (k^{j+1}) = \sigma (x) \neq 0 \). But
THE RIGIDITY PROBLEM FOR STABLE SPACES

\[ \Sigma X \] is \( i \)-connected and \( X_\alpha = K(\pi_\alpha(X), i) \), so every \( \alpha \) is null-homotopic.

**Remarks.** 1. Combining [3] with this Corollary, it is not hard to show that if \( X \) is in \( S_1 \) and every \( k \)-invariant is obtained from the fundamental class by a primary cohomology operation, then \( X \) is not rigid.

2. It would be interesting to get information on \( G(X) \) for spaces \( X \) with 3 or more homotopy groups. If \( X \) is rigid, then there is some \( j \) with \( G(X_j) \) nontrivial, yet \( G(X_{j+1}) \) is trivial.

We now compare these results with the category of finite complexes \( S_2 \).

**Proposition 3.** If \( X \) is a nontrivial, finite, path-connected complex, the group of stable self-equivalences is nontrivial. (That \( X \) is nontrivial means \( X \) does not have the homotopy-type of a point.)

**Proof.** D. Anderson and M. Barratt have proved (unpublished) that the additive stable order of \( 1_X \) cannot be 2. (This may be shown by looking at the Steenrod operation \( Sq^1 \) in \( X \wedge X \). If the stable order of \( 1_X \) is 2, the same would be true for \( 1_{X \wedge X} \). Taking \( u \in H^i(X : \mathbb{Z}_2) \), with \( Sq^1u \neq 0 \), of maximal dimension, we get \( Sq^2(u \wedge u) \neq 0 \), which quickly leads to a contradiction. Or one may study top dimensional cells of \( X \), and copy the proof (as in [1]) that the homotopy groups of \( S^n \cup 2 \mathbb{E}^{n+1} \) have an element of order 4.

Then just as before, \( 1_X \) and \(-1_X \) represent different elements in our group.

**Proposition 4.** Let \( Y \) be a connected, finite complex \((i - 1)\)-connected, \( i > 1 \), and \( \dim Y < 2i - 1 \). Let \( Y_j \) be a Postnikov term for \( Y \), \( i < j < 2i - 1 \).

If \( Y_j \) is rigid, then \( j < \dim(Y) \).

**Proof.** By Proposition 3, it suffices to show that if \( j > \dim(Y) + 1 \), \( G(Y) = G(Y_j) \).

Clearly, any two homotopy equivalences, which represent the same class on \( Y_j \), must be homotopic, for \( Y \) and \( Y_j \) have the same \( j \)-type.

On the other hand, the fibration \( p: Y \rightarrow Y_j \) has a section over the \((j + 1)\)-skeleton \( S: Y^{(j+1)}_j \rightarrow Y_j \). If \( f: Y_j \rightarrow Y_j \) is a (cellular) homotopy equivalence, we may follow \( Y \rightarrow Y_j \rightarrow Y_j \) by \( S \), to get a homotopy equivalence of \( Y \) to itself, which maps to the class of \( f \). Thus the natural map \( G(Y) \rightarrow G(Y_j) \) is also onto.

**Remark.** On the negative side of things, this paper shows that nonrigidity is almost everywhere in stable homotopy, and the consequences—such as a plurality of possible \( k \)-invariants for a space—cannot be avoided.

**References**


SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455