THE RIGIDITY PROBLEM FOR STABLE SPACES

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Abstract. This note treats the problem of when the group of homotopy self-equivalences of a space is trivial. For stable spaces, with finitely many nonvanishing homotopy groups, we give a complete solution in an inductive sense. One of the consequences of this result is that for any stable space, with precisely two nonvanishing homotopy groups, the group of self-equivalences is nontrivial.

If $X$ is a (pointed) path connected space, we denote by $G(X)$ the group of homotopy classes of homotopy equivalences from $X$ to itself (all preserving base point). This is the homotopy theorist's analogue of the group of automorphisms of a group, and it has now been studied by many authors. For example, C. Wilkerson [8] and D. Sullivan [7] have shown that when $X$ is a simply-connected finite complex, $G(X)$ is a finitely-presented group. Their methods, refining methods in [4], use algebraic groups, and they are not fine enough to get closer to $G(X)$ than a subgroup of finite index, or a quotient by a finite, normal subgroup. On the other hand, there is a space $X$, which at every prime but 3 is a $K(Z, 4)$, but for which $G(X)$ is trivial [5]. Since $G(K(Z, 4)) = \mathbb{Z}_2$, the integers mod 2, we see that a space $X$ may be equivalent to a space $Y$, at all but one prime, yet $G(X)$ is trivial and $G(Y)$ is not. We shall call a space $X$, for which $G(X)$ is trivial, rigid, and we propose to study such spaces--in a stable sense--in this paper.

We shall work in a category whose objects are spaces with finitely-many nonvanishing homotopy groups, in a stable range. Specifically, such a space $X$ shall be endowed with a finite Postnikov tower

$$X = X_k \overset{\pi_k}{\rightarrow} \cdots \overset{\pi_n}{\rightarrow} X_n$$

where $\pi_i$ is a principal fibration with fibre $K(\pi_i(X), i), n > 1, \text{ and } k < 2n - 1$. Such a space is infinitely de-loopable, and our continuous maps and homotopies shall be presumed to be infinitely de-loopable also. Base points are assumed but omitted from the notation. We denote this category by $\mathcal{S}_1$. For comparison purposes--at the end of this note--we denote by $\mathcal{S}_2$ the category of simply-connected finite complexes, and stable (in the sense of suspension) homotopy classes of maps.

In a given Postnikov system, if $j > i$, there is a natural homomorphism (see [2])

$$\phi_{i,j}: G(X_j) \rightarrow G(X_i),$$

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which is studied in [3]. The image of $\phi_{m+1,m}$ is precisely known, while the kernel is there related to certain cohomology groups and extensions.

Our idea is now to study the rigidity problem, that is when a space is rigid, by induction. We easily characterize when $K(\pi, n), n > 1,$ is rigid (Proposition 2), and then go on to give necessary and sufficient conditions for $\ker(\phi_{m+1,m})$ to be trivial. As a corollary, we may conclude that there is no space $X$ in $\mathbb{S}_1,$ with precisely two nonvanishing homotopy groups, which is rigid. In conclusion, we compare this situation with that in the category $\mathbb{S}_2.$ We recall (Proposition 3) the well-known result that there is no nontrivial $X$ in $\mathbb{S}_2,$ which is rigid. We then show (Proposition 4) that if $X_k$ is a Postnikov term for a finite stable complex $X,$ and $X_k$ is rigid, then $\dim(X) > k.$

Finally, we would like to point out that our results for $\mathbb{S}_1$ do not require the homotopy groups of the spaces in question to be finitely-generated, as opposed to many of the other papers in the field. For example, our Proposition 2 actually holds for all Abelian groups $\pi,$ if we choose $X$ to be cellular.

**Proposition 1.** If $X$ is a rigid space, in the category $\mathbb{S}_1,$ then the identity map $1_X$ has order 2 in the sense of loop multiplication in $X.$ In particular, every element in the homotopy or stable homology groups of $X$ has order 2.

**Proof.** Let $[X, X]$ denote the Abelian group of homotopy classes of maps from $X$ to itself, under loop multiplication. Assume that the order of the class of $1_X$ is different from 2. Then $-1_X$ is not homotopic to $1_X.$ As $-1_X$ is clearly a homotopy equivalence, $G(X)$ has at least two elements.

For the rest of the proposition, we simply observe that the addition in $[X, X]$ acts additively on homotopy and stable homology (that is homology groups in the stable range).

**Proposition 2.** Let $X$ be a space in $\mathbb{S}_1,$ with a single, nonvanishing homotopy group, i.e., $X = K(\pi, n), n > 1.$ Then $X$ is rigid precisely when $\pi = \mathbb{Z}_2,$ the integers mod 2.

**Proof.** If $X$ is rigid, then we know that every element of $\pi$ has order 2. I claim that $\pi$ is a $\mathbb{Z}_2$-vector space. It is trivial to construct a group homomorphism from a $\mathbb{Z}_2$-vector space $V$ onto $\pi,$ say $g: V \to \pi.$ But the kernel of $g$ is trivially checked to be a vector subspace, proving that $\pi$ is the quotient of a vector space by a subspace, and hence, $\pi$ is itself a vector space.

Now, it is well known that $G(K(\pi, n)) = \text{Aut}(\pi),$ the group of automorphisms of $\pi.$ But the only $\mathbb{Z}_2$-vector space, without a nontrivial automorphism, has dimension 1.

Before stating our main theorem, it will be convenient to specify some notation.

a. $\Sigma X$ means the (reduced) suspension of $X.$ $\Sigma f$ means the suspension of the map $f. \sigma: H^p(Y) \cong H^{p+1}(\Sigma Y)$ is the usual suspension isomorphism in cohomology. $\Omega X$ means loops on $X.$
b. If $Y$ is $(n - 1)$-connected, we shall freely write $i_n \in H^n(Y; \pi_n(Y))$ for the fundamental class. If $x \in H^m(Y; \pi)$, write $\phi_x: Y \to K(\pi, m)$ for a map such that $\phi_x^*(i_m) = x$.

c. For a fibration in a given Postnikov tower, say

$$X_{n+1} \xrightarrow{\pi_{n+1}} X_n,$$

we shall abbreviate the fibre $K(\pi_{n+1}(X), n + 1)$ to $K$. We write the inclusion of the fibre $i: K \to X_{n+1}$.

**Theorem.** Let $X$ be a space in $S_1$, and let

$$\phi_{m+1,m}: G(X_{m+1}) \to G(X_m)$$

be the homomorphism described above. Then a necessary and sufficient condition that $\ker(\phi_{m+1,m})$ be trivial, is that for every cohomology class

$$x \in H^{m+1}(X_{m+1}; \pi_{m+1}(X)),$$

for which $i \cdot \phi_x + 1_{X_{m+1}}$ induces an isomorphism on homology in dimension $m + 1$, there is a map

$$\alpha: \Sigma X_{m+1} \to X_m$$

so that $\alpha^*(k^{m+2}) = \sigma(x)$, with $k^{m+2}$ being the $k$-invariant of the fibration.

**Proof.** We are working in the stable category $S_1$. If we suppose $(f) \in \ker(\phi_{m+1,m})$, then the diagram

$$\begin{array}{ccc}
X_{m+1} & \xrightarrow{f} & X_{m+1} \\
\downarrow & \searrow & \downarrow \\
X_m & \xrightarrow{0} & X_m + 1
\end{array}$$

is homotopy commutative (0 sends all $X_{m+1}$ to the base point).

By exactness, there is a map $\psi: X_{m+1} \to K = \pi_{m+1}^{-1}(pt.)$, so that

$$\begin{array}{ccc}
K & \xrightarrow{i} & X_{m+1} \\
\downarrow & \searrow & \downarrow \\
X_{m+1} & \xrightarrow{f} & X_{m+1}
\end{array}$$

is homotopy commutative.

Now, the fibration $\pi_{m+1}$ gives rise to an exact sequence of spaces

$$\cdots \to \Omega X_m \xrightarrow{\gamma} K \xrightarrow{i} X_{m+1} \xrightarrow{\pi_{m+1}} X_m \to \cdots ;$$

it is well known that if we identify cohomology of $\Sigma \Omega X_m$ and $X_m$ in the stable range, then $\sigma(\gamma^*(i_{m+1})) = k^{m+2}$, the $k$-invariant.

By exactness, the map $f - 1_{X_{m+1}} \simeq i \cdot \psi$ will be null-homotopic (and hence $f \simeq 1_{X_{m+1}}$) precisely when there is a map $\alpha': X_{m+1} \to \Omega X_m$, with
If we suspend this diagram and write $\alpha = \Sigma \alpha'$, then stably we get a homotopy commutative diagram

$$
\begin{array}{ccc}
\Sigma X_{m+1} & \xrightarrow{\alpha} & X_m \\
\Sigma \psi_f & \downarrow{\Sigma \gamma} & \\
K(\pi_{m+1}(X), m + 2)
\end{array}
$$

On cohomology, $\alpha^* (k^{m+2}) = (\Sigma \psi_f)^*(i_{m+2}) = \sigma(\psi_f^*(i_{m+1}))$.

To complete the proof, we must see that the classes $x_i$ with the condition given in the statement of the theorem, are precisely those classes of the form $\psi_f^*(i_{m+1})$. But given such a class $x_i$, $i \cdot \phi_x + 1_{x_{m+1}}$ clearly induces isomorphisms on homology in dimensions through $m + 2$ (recall $H_{m+2}(K) = 0$) and is thus a homotopy equivalence $f$. Then $f - 1_{x_{m+1}} \simeq i \cdot \phi_x$, so we take $\phi_x = \psi_f$ and $\psi_f^*(i_{m+1}) = \phi_x^*(i_{m+1}) = x_i$. On the other hand, given $\psi_f$ for a homotopy equivalence $f$, with $\{f\} \in \ker(\phi_{m+1,m})$—the class $\psi_f^*(i_{m+1})$ clearly meets the condition of the theorem.

We remark that the theorem is clearly valid for $k(\phi_{j,i})$, $j > i$, when the homotopy groups in dimensions between $i$ and $j$ vanish.

**Corollary.** If $X$ is a space in $\mathbb{S}_1$ with precisely two nonvanishing homotopy groups, $X$ is not rigid.

**Proof.** By our propositions, we assume the two nonvanishing groups $\pi_i(x)$ and $\pi_j(x)$, $i < j < 2i - 1$, are both direct sums of copies of $\mathbb{Z}_2$. We shall prove $\ker(\phi_{j,i})$ is nontrivial.

By [6], the $\mathbb{Z}_2$-cohomology of $K(\mathbb{Z}_2, i)$ is nonzero in every dimension not less than $i$. In particular, the subgroup

$$\mathbb{H}^i\left(K(\pi_i(X), i); \pi_j(X)\right) \subseteq \mathbb{H}^i\left(X_j; \pi_j(X)\right)$$

is nontrivial. Let $x \neq 0$ be a class in this subgroup. Consider the map

$$X_j = X \xrightarrow{\phi_x \times 1_X} K(\pi_j(X), j) \times X \xrightarrow{\bar{\mu}} X,$$

where $\bar{\mu}$ is the action of the fibre. It clearly induces isomorphisms on homotopy groups and is thus a homotopy equivalence. Stably, it is just $i \cdot \phi_x + 1_X$.

We must ask whether there is $\alpha$; $\Sigma X \to X_i$ with $\alpha^* (k^{i+1}) = \sigma(x) \neq 0$. But...
\( \Sigma X \) is \( i \)-connected and \( X_i = K(\pi_i(X), i) \), so every \( \alpha \) is null-homotopic.

Remarks. 1. Combining [3] with this Corollary, it is not hard to show that if \( X \) is in \( S_1 \) and every \( k \)-invariant is obtained from the fundamental class by a primary cohomology operation, then \( X \) is not rigid.

2. It would be interesting to get information on \( G(X) \) for spaces \( X \) with 3 or more homotopy groups. If \( X \) is rigid, then there is some \( j \) with \( G(X_j) \) nontrivial, yet \( G(X_{j+1}) \) is trivial.

We now compare these results with the category of finite complexes \( S_2 \). The following is well known.

Proposition 3. If \( X \) is a nontrivial, finite, path connected complex, the group of stable self-equivalences is nontrivial. (That \( X \) is nontrivial means \( X \) does not have the homotopy-type of a point.)

Proof. D. Anderson and M. Barratt have proved (unpublished) that the additive stable order of \( 1_X \) cannot be 2. (This may be shown by looking at the Steenrod operation \( Sq^1 \) in \( X \wedge X \). If the stable order of \( 1_X \) is 2, the same would be true for \( 1_{X \wedge X} \). Taking \( u \in H^i(X : Z_2) \), with \( Sq^1u \neq 0 \), of maximal dimension, we get \( Sq^2(u \wedge u) \neq 0 \), which quickly leads to a contradiction. Or one may study top dimensional cells of \( X \), and copy the proof (as in [1]) that the homotopy groups of \( S^n \cup_2 e^{n+1} \) have an element of order 4).

Then just as before, \( 1_X \) and \( -1_X \) represent different elements in our group.

Proposition 4. Let \( Y \) be a connected, finite complex \((i - 1)\)-connected, \( i > 1 \), and \( \dim Y < 2i - 1 \). Let \( Y_j \) be a Postnikov term for \( Y \), \( i < j < 2i - 1 \).

If \( Y_j \) is rigid, then \( j < \dim(Y) \).

Proof. By Proposition 3, it suffices to show that if \( j > \dim(Y) + 1 \), \( G(Y) = G(Y_j) \).

Clearly, any two homotopy equivalences, which represent the same class on \( Y_j \), must be homotopic, for \( Y \) and \( Y_j \) have the same \( j \)-type.

On the other hand, the fibration \( p: Y \to Y_j \) has a section over the \((j + 1)\)-skeleton \( S: Y_j^{(j+1)} \to Y \). If \( f: Y_j \to Y_j \) is a (cellular) homotopy equivalence, we may follow \( Y \to Y_j \to Y_j \) by \( S \), to get a homotopy equivalence of \( Y \) to itself, which maps to the class of \( f \). Thus the natural map \( G(Y) \to G(Y_j) \) is also onto.

Remark. On the negative side of things, this paper shows that nonrigidity is almost everywhere in stable homotopy, and the consequences—such as a plurality of possible \( k \)-invariants for a space—cannot be avoided.

References


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