PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 1, June 1979

A THEOREM ON EXTENDING REPRESENTATIONS

RAYMOND C. FABEC[1]

ABSTRACT. We obtain sufficient conditions for a unitary representation of a closed subgroup M of a separable locally compact group G to have a unique unitary extension to G. The conditions depend on the behavior of the representation on a closed normal subgroup N of G contained in M. We then discuss an application of the theorem in particle physics.

Let N be a closed normal subgroup of a separable locally compact group G, and let M be a closed subgroup of G containing N. Let $H = \{m \in M: gmg^{-1} \in M \text{ for } g \in G\}$. Assume L is a representation of M such that $L|_N$ is type I, and $L|_N = \int_{\hat{N}}^{\oplus} n(\alpha)\alpha \, d\mu(\alpha)$, μ a standard measure on $\hat{N}$; i.e. there is a μ conull set in $\hat{N}$ which is standard. By the von Neumann selection theorem, there is a cross-section c from $\hat{N}$ into the concrete irreducible representations of N such that c is Borel on a conull set in $\hat{N}$. Then

$$\int_{\hat{N}}^{\oplus} n(\alpha)\alpha \, d\mu(\alpha) = \int_{\hat{N}}^{\oplus} n(\alpha)c(\alpha) \, d\mu(\alpha),$$

and this gives the primary decomposition of $L|_N$. G acts on $\hat{N}$ by $\alpha \cdot g = \{R \cdot g: R \in \alpha\}$ where $(R \cdot g)(n) = R(gng^{-1})$. We do the transitive case first.

THEOREM 1. *Assume μ is supported on $\alpha_0 \cdot G$. Assume the stabilizer K of α_0 is contained in H and $L|_H \cdot g$ is unitarily equivalent to $L|_H$ for each $g \in G$. Then L has a unique unitary extension to G. In fact, there is a representation W of K such that* $\operatorname{ind}_K^G W$ *is the extension. This extension is irreducible precisely when W is irreducible.*

REMARK. $L = \int_{G/M}^{\oplus} \operatorname{ind}_{x^{-1}Kx}^{M} W \cdot x \, d\bar{\mu}(x)$ where $\bar{\mu}$ is a quasi-invariant measure on G/M.

PROOF. Let P be the projection valued measure based on the Borel subsets of $\alpha_0 \cdot G \simeq K \setminus G$ giving the direct integral decomposition $L|_N = \int^{\oplus} n(\alpha)c(\alpha) \, d\mu(\alpha)$. Since $L|_H \cdot g \simeq L$ for $g \in G$, $L|_N \cdot g \simeq L|_N$ for $g \in G$; and the argument given by Mackey [4, p. 296], shows μ is quasi-invariant and $n(\alpha \cdot g^{-1}) = n(\alpha)$ for μ-a.e. α for each g. Hence, since G acts transitively, n is essentially constant. Therefore

$$L|_N \simeq n\int^{\oplus} c(\alpha) \, d\mu(\alpha) = n\int^{\oplus} c(\alpha) \, d\mu(\alpha) = n\int^{\oplus} \omega \cdot \gamma(Kx) \, d\mu(Kx)$$

Received by the editors February 27, 1978 and, in revised form, July 6, 1978.
AMS (MOS) subject classifications (1970). Primary 22D10, 22D30; Secondary 22E70.
[1]Research supported in part by a Louisiana State Summer Faculty Research Award.

where $\omega \in \alpha_0$ and $\gamma\colon K \setminus G \to G$ is a Borel cross-section.

Since $L_m L|_N L_m^{-1} = L|_N \cdot m \simeq L|_N$ and $n\int^{\oplus}\omega \cdot \gamma(Kx)\, d\mu(Kx)$ is the central decomposition of $L|_N$, $L_m P(E) L_m^{-1} = P(E \cdot m^{-1})$ for Borel subsets E of $K \setminus G$ and $m \in M$. Furthermore, the orbit space $(K \setminus G)^M$ of $K \setminus G$ under M is isomorphic to G/M. In fact $KxM = xx^{-1}KxM = xM$, for $x^{-1}Kx \subset x^{-1}Hx = H$. Hence the orbit space is standard; and if $\bar{\mu}$ is the image of μ under the map $Kx \mapsto xM$, μ decomposes uniquely $\bar{\mu}$-a.e. into an "integral of meaures", $\mu = \int_{G/M} \mu_{xM} d\bar{\mu}(xM)$, where each μ_{xM} is a Borel measure on $K \setminus G$ with $\mu_{xM}(K \setminus G - KxM) = 0$ (see [2, Theorem 11]). Since μ is quasi-invariant under M, μ_{xM} is quasi-invariant under M for $\bar{\mu}$-a.e. xM.

The decomposition $\mu = \int_{G/M} \mu_{xM}\, d\bar{\mu}(xM)$ yields a decomposition of P into $\int_{G/M}^{\oplus} P(xM)\, d\bar{\mu}(xM)$ where each $P(xM)$ is a projection valued measure defined on the Borel subsets of xM satisfying $P(xM)(E) = 0$ iff $\mu_{xM}(E) = 0$. Define a projection valued measure Q on the Borel subsets of G/M by $Q(F) = P(\bigcup F)$. Then

$$L_m^{-1} Q(F) L_m = P\Big(\bigcup F \cdot m\Big) = P\Big(\bigcup F\Big) = Q(F).$$

Therefore L decomposes into a direct integral over G/M; namely, $L = \int^{\oplus} L(xM)\, d\bar{\mu}(xM)$. Furthermore,

$$L(xM)|_N = n\int_{KxM}^{\oplus} \omega \cdot \gamma(Kz)\, d\mu_{xM}(Kz)$$

and

$$L(xM)(m^{-1})P(xM)(E)L(xM)(m) = P(xM)(E \cdot m),$$

E any Borel subset of KxM. Hence the pair $(L(xM), P(xM))$ forms a transitive system of imprimitivity based on the transitive M space KxM. Let $\sigma\colon G/M \to G$ be a Borel cross-section. By Mackey's imprimitivity theorem [3, Theorem 6.6], $L(xM) = \operatorname{ind}_{\sigma(xM)^{-1}K\sigma(xM)}^{M} W(xM)$, where $W(xM)$ is a unitary representation of $\sigma(xM)^{-1}K\sigma(xM)$. One can show $W(xM)|_N \simeq n\omega \cdot \sigma(xM)$.

We consider the representation $L|_H$. The orbit space $(K \setminus G)^H$ of $K \setminus G$ under H is isomorphic to $H \setminus G$ since $KxH = KxHx^{-1}x = Hx$. Hence $\mu = \int \nu_{Hx} d\bar{\nu}(Hx)$ where $\bar{\nu}$ is the image of μ under the map $Kx \to Hx$ and ν_{Hx} is an H quasi-invariant measure on Hx. This decomposition, as before, gives a decomposition of P into $\int^{\oplus} P(Hx)\, d\bar{\nu}(Hx)$.

$\bar{\mu}$ is the image of $\bar{\nu}$ under the map $Hx \to xM$. Therefore $\bar{\nu}$ decomposes relative to $\bar{\mu}$: $\bar{\nu} = \int_{G/M} \bar{\nu}_{xM}\, d\bar{\mu}(xM)$, $\bar{\nu}_{xM}$ a measure on $H \setminus G$ with

$$\bar{\nu}_{xM}(H \setminus G - xM) = 0.$$

Hence

$$\mu = \int_{H\setminus G} \nu_{Hx}\, d\bar{\nu}(Hx) = \int_{G/M}\int_{H\setminus G} \nu_{Hy}\, d\bar{\nu}_{xM}(Hy)\, d\bar{\mu}(xM)$$

$$= \int_{G/M} \mu_{xM}\, d\bar{\mu}(xM).$$

By uniqueness of decompositions, $\mu_{xM} = \int_{H\backslash G} \nu_{Hy}\, d\bar{\nu}_{xM}(Hy)$ for $\bar{\mu}$-a.e. xM. Furthermore, one has $P(xM) = \int_{H\backslash G} P(Hy)\, d\bar{\nu}_{xM}(Hy)$.

Define a projection valued measure Q^* on the Borel subsets of $H \backslash G$ by $Q^*(F) = P(\bigcup F)$. Then $L_h Q^*(F) L_h^{-1} = Q^*(F)$ for $h \in H$. Therefore $L|_H$ decomposes into a direct integral:

$$L|_H = \int_{H\backslash G}^{\oplus} L(Hy)\, d\bar{\nu}(Hy),$$

where $L(Hy)|_N = n\int_{Hy} \omega \cdot \gamma(Kz)\, d\nu_{Hy}(Kz)$. Furthermore, since $P(xM) = \int_{xM} P(Hy)\, d\bar{\nu}_{xM}(Hy)$, one has

$$L(xM)|_H = \operatorname*{ind}_{\sigma(xM)^{-1}K\sigma(xM)}^{M} W(xM)|_H = \int^{\oplus} L(Hy)\, d\bar{\nu}_{xM}(Hy).$$

This decomposition of $(L(xM), P(xM))$ is precisely the one given by Mackey's subgroup theorem. Note KxM as an M space is isomorphic to $\sigma(xM)^{-1}K\sigma(xM) \backslash M$ and the orbit space of KxM under H is isomorphic to the double coset space of $\sigma(xM)^{-1}K\sigma(xM) - H$ double cosets in M. The correspondence with KxM carries the double coset $\sigma(xM)^{-1}K\sigma(xM)mH$ to $K\sigma(xM)mH = H\sigma(xM)m$. Hence the double cosets of KxM are the H cosets in KxM. Now let ρ: $H \backslash G \to G$ be a Borel cross-section. Then $\rho|_{KxM} \to G$ is a cross-section of the $K - H$ double cosets in KxM and $Hy = K\sigma(xM)\sigma(xM)^{-1}\rho(Hy)H$. Therefore

$$L(Hy) \simeq \operatorname*{ind}_{\rho(Hy)^{-1}K\rho(Hy)}^{H} W(xM) \cdot \sigma(xM)^{-1}\rho(Hy)$$

$\bar{\nu}_{xM}$-a.e. Hy $\bar{\mu}$-a.e. xM. For more details, see the proof of the subgroup theorem [**4**, p. 227].

We now have

$$L|_H \simeq \int_{H\backslash G}^{\oplus} \operatorname*{ind}_{\rho(Hy)^{-1}K\rho(Hy)}^{H} W(yM) \cdot \sigma(yM)^{-1}\rho(Hy)\, d\bar{\nu}(Hy)$$

and $P = \int^{\oplus} P(Hy)\, d\bar{\nu}(Hy)$ where $P(Hy)$ is the canonical projection valued measure associated with the induced representation $L(Hy)$. Therefore

$$L|_H \simeq L|_H \cdot g \simeq \int^{\oplus} \operatorname*{ind}_{g^{-1}\rho(Hy)^{-1}K\rho(Hy)g}^{H} W(yM) \cdot \sigma(yM)^{-1}\rho(Hy)g\, d\bar{\nu}(Hy).$$

Since $\rho(Hy)g = \rho(Hyg)h$ for some h,

$$L|_H \simeq \int^{\oplus} \operatorname*{ind}_{\rho(Hyg)^{-1}K\rho(Hyg)}^{H} W(yM) \cdot \sigma(yM)^{-1}\rho(Hyg)\, d\bar{\nu}(Hy).$$

In this last direct integral decomposition, P decomposes into $P = \int^{\oplus} P(Hy) \cdot g\, d\bar{\nu}(Hy)$ where $P(Hy) \cdot g(E) = P(Hy)(E \cdot g^{-1})$. Changing $y \to yg^{-1}$ in the last decomposition yields

$$L|_H \simeq \int^{\oplus} \operatorname*{ind}_{\rho(Hy)^{-1}K\rho(Hy)}^{H} W(yg^{-1}M) \cdot \sigma(yg^{-1}M)^{-1}\rho(Hy)\, d\bar{\nu}(Hy)$$

and

$$P = \int^{\oplus} P(Hyg^{-1}) \cdot g \, d\bar{\nu}(Hy).$$

But

$$P(Hyg^{-1}) \cdot g = P(Hy).$$

Hence we have two decompositions of $(L|_H, P)$ relative to Q^*. Therefore

$$\left(\operatorname{ind} W(yg^{-1}M) \cdot \sigma(yg^{-1}M)^{-1}\rho(Hy), P(Hy)\right)$$
$$\simeq \left(\operatorname{ind} W(yM) \cdot \sigma(yM)^{-1}\rho(Hy), P(Hy)\right)$$

for $\bar{\nu}$-a.e. Hy. It follows that

$$W(yg^{-1}M) \cdot \sigma(yg^{-1}M)\rho(Hy) \simeq W(yM) \cdot \sigma(yM)^{-1}\rho(Hy)$$

for $\bar{\nu}$-a.e. Hy for each $g \in G$. Hence there is a representation W of K such that $W(xM)$ is unitarily equivalent to the representation $\sigma(xM)^{-1}k\sigma(xM) \to W(k)$ for $\bar{\mu}$-a.e. xM. We then have

$$L \simeq \int_{G/M}^{\oplus} \operatorname*{ind}_{\sigma(xM)^{-1}K\sigma(xM)}^{M} W \cdot \sigma(xM) \, d\bar{\mu}(xM).$$

Let $U = \operatorname{ind}_K^G W$. By Mackey's subgroup theorem, U extends L. U is unique for if U is an extension of L, (U, P) is a transitive system of imprimitivity based on $K \setminus G$. Hence $U = \operatorname{ind}_K^G W'$. But then

$$\left(\operatorname*{ind}_{\sigma(xM)^{-1}K\sigma(xM)}^{M} W \cdot \sigma(xM), P(xM)\right)$$
$$\simeq \left(\operatorname*{ind}_{\sigma(xM)^{-1}K\sigma(xM)}^{M} W' \cdot \sigma(xM), P(xM)\right)$$

for $\bar{\mu}$-a.e. xM. Hence $W \simeq W'$.

The commuting rings of U and W are isomorphic. Hence U is irreducible when W is. Q.E.D.

REMARK. The stabilizer K of α_0 is contained in H if the stabilizer of μ-a.e. α is contained in M.

We next assume μ is based on a standard G-invariant subset S of $\hat{N}$.

THEOREM 2. *Assume μ is based on S and G acts regularly on S and $L|_H \cdot g \simeq L|_H$ for $g \in G$. Then L has a unique extension from M to G provided that the stabilizer of α is contained in M for μ-a.a. α.*

PROOF. Let S^G be the orbit space for S under G. Let $\bar{\mu}$ be the quotient measure. The action is regular if there exists a $\bar{\mu}$ conull set $\bar{S}_0$ in S^G such that $\bar{S}_0$ is standard. Replacing S by the inverse image S_0 of $\bar{S}_0$ in S, we may

assume S^G is standard. Let P be the projection valued measure defined in the proof of Theorem 1. The measure μ decomposes relative to $\bar{\mu}$; $\mu = \int \mu_{\bar{s}}\, d\bar{\mu}(\bar{s})$ where each $\mu_{\bar{s}}$ is a G quasi-invariant measure based on $\bar{s}$. This leads to a decomposition of both P and L: $P = \int^{\oplus} P^{\bar{s}}\, d\bar{\mu}(\bar{s})$ and $L = \int^{\oplus} L^{\bar{s}}\, d\bar{\mu}(\bar{s})$. Since the stabilizer of α is contained in M for μ-a.a. $\alpha \in S$, the stabilizer of $\alpha \in \bar{s}$ is contained in M for $\mu_{\bar{s}}$-a.a. α for $\bar{\mu}$-a.a. $\bar{s}$. Since $L|_H \cdot g \simeq L|_H$, $L^{\bar{s}}|_H \cdot g \simeq L^{\bar{s}}|_H$ for $\bar{\mu}$-a.e. $\bar{s}$. By Fubini's theorem, for $\bar{\mu}$-a.e. $\bar{s}$, $L^{\bar{s}}|_H \cdot g \simeq L^{\bar{s}}|_H$ a.e. g. But $\{g\colon L^{\bar{s}}|_H \cdot g \simeq L^{\bar{s}}|_H\}$ is a subgroup of G. Hence for $\bar{\mu}$-a.e. $\bar{s}$, $L^{\bar{s}}|_H$ is invariant under G. Hence we may apply Theorem 1 and obtain an extension $R^{\bar{s}}$ of $L^{\bar{s}}$ to G for $\bar{\mu}$-a.a. $\bar{s}$. The extension is a Borel function of $\bar{s}$ and hence $\int^{\oplus} R^{\bar{s}} d\bar{\mu}(\bar{s})$ is an extension of L. Since $R^{\bar{s}}$ is unique a.e. $\bar{s}$, this extension is unique. Q.E.D.

REMARK. These theorems remain valid for σ-representations. One can either work in the group extension or modify the above proofs.

As an application of the theorem we show how it is possible to determine the dynamics of a free particle system having time translations which do not commute with spatial translations. For instance the spatial transformations of $\mathbf{R}^3$ may be the Euclidean group $E_3 = \{(x, R)\colon x \in \mathbf{R}^3 \text{ and } R \in \mathrm{SO}(3)\}$ with multiplication defined by $(x, R)(y, S) = (x + Ry, RS)$. Time translation by t, however, will have the property $t(x, R) = (e^t x, R)t$. This may occur, for instance, if physical space expands exponentially with time. In any case, for this group a free particle of spin j, j a half integer, is a projective representation U such that $U_{(x,R)}f(y) = e^{2\pi i x\cdot y} D^j(R) f(R^{-1}y)$ for $f \in L^2(\mathbf{R}^3, C^{2j+1})$ where D^j is the projective irreducible representation of SO(3) with dimension $2j + 1$. We shall, for convenience, assume j is an integer, for in this case D^j is an ordinary representation. Next note one has $M = E_3$ and $N = \mathbf{R}^3$. Then

$$U|_{\mathbf{R}^3} = (2j + 1)\int_{\mathbf{R}^3}^{\oplus} e^{2\pi i x\cdot(\,)}\, dx$$

and since

$$\exp(2\pi i x \cdot (\,)) \cdot t(0, R) = \exp\bigl(2\pi i e^{-t} R^{-1} x \cdot (\,)\bigr),$$

the orbit of $e_1 = (1, 0, 0)$ is almost all of $\mathbf{R}^3$. The stabilizer K of e_1 is $\{(y, R)\colon Re_1 = e_1,\ y \in \mathbf{R}^3\}$ while $H = M = E_3$. Hence Theorem 1 applies and we need only find W. Note $U|_{E_3} = \int_{\mathbf{R}}^{\oplus} W(r)\, dr$ where $W(r)$ is the representation of E_3 on $L^2(S_r, m_r, C^{2j+1})$, m_r the Lebesgue measure on the sphere of radius e^r, defined by

$$W(r)(t, R)f(s) = e^{2\pi i x\cdot s} D^j(R) f(R^{-1}y).$$

One can show $W(r) = \mathrm{ind}_K^M (W \cdot t) = \mathrm{ind}_{tKt^{-1}}^M W \cdot t$ where $W(x, R) = e^{2\pi i x_1} D^j(R)$ for $(x_1, x_2, x_3, R) \in K$. Hence $U = \mathrm{ind}_K^G W$.

For more information on particle systems having these symmetries, see [1].

REFERENCES

1. R. Fabec, *Free particle systems and their dynamics in de Sitter space*, J. Mathematical Phys. **18** (1977), 2087–2091.

2. G. W. Mackey, *Induced representations of a locally compact group*. I, Ann. of Math. **2** (1952), 101–139.

3. _____, *The theory of unitary group representations*, 1st ed., Univ. of Chicago Press, Chicago, Ill., 1976, 372 pp.

4. _____, *Unitary representations of group extensions*. I, Acta Math. **99** (1958), 265–311.

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803