PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 1, June 1979

ANOSOV DOES NOT IMPLY INFINITESIMALLY ERGODIC

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ABSTRACT. A class of hyperbolic toral automorphisms is exhibited which are not infinitesimally ergodic. These provide counterexamples to the conjecture that Anosov diffeomorphisms are infinitesimally ergodic.

In [**2**, p. 944], J. Robbin conjectured that Anosov diffeomorphisms are infinitesimally ergodic. In this note we disprove the conjecture by showing the existence of a class of hyperbolic toral automorphisms which fail to be infinitesimally ergodic.

Let $G: M \to M$ denote a diffeomorphism of the compact manifold M. The *adjoint representation* $G^{\#}$ is the operator on vector fields X given by

$$G^{\#}X = TG \circ X \circ G^{-1}.$$

If ω denotes a G-invariant measure on M, let $H^1(TM) = H^1(TM, \omega)$ denote the Sobolev space of vector fields with one square-integrable weak derivative. G is *infinitesimally ergodic* if the bounded operator

$$I - G^{\#}: H^1(TM) \to H^1(TM)$$

has dense range. The adjoint

$$(G^{\#})^*: H^{-1}(T^*M) \to H^{-1}(T^*M)$$

is the continuous extension of the operator "pull-back by G" on smooth differential forms.

A unimodular matrix with integer entries and spectrum off the unit circle projects to a diffeomorphism of $T^n = R^n/Z^n$ called a hyperbolic toral automorphism.

THEOREM. *If A is a hyperbolic toral automorphism with two distinct eigenvalues inside* (*outside*) *the unit circle, then A is not infinitesimally ergodic.*

PROOF. We construct a nonzero $\eta \in H^{-1}(T^*M)$ which lies in the kernel of $I - (A^{\#})^*$. Regarding elements $\eta \in H^{-1}(T^*M)$ as n-tuples of functions (in $H^{-1}(M)$) we express η as a vector Fourier series

$$\eta(\theta) = \sum \hat{\eta}(\mu)e^{i(\theta\cdot\mu)}$$

where $\mu \in Z^n$ and $\hat{\eta}(\mu) \in \mathbf{C}^n$. As $(A^{\#})^*\eta = A^t \circ \eta \circ A$ (A^t denotes the transpose), η lies in the kernel of $I - (A^{\#})^*$ when its Fourier coefficients

Received by the editors August 14, 1978.

AMS (MOS) subject classifications (1970). Primary 58F15.

Key words and phrases. Anosov diffeomorphism, hyperbolic toral automorphism, ergodic, Sobolev space, adjoint representation.

0002-9939/79/0000-0285/$01.50

satisfy

$$A^t\hat{\eta}\left((A^t)^{-1}\mu\right) = \hat{\eta}(\mu). \tag{1}$$

For simplicity assume A has distinct eigenvalues. Then $B = (A^t)^{-1}$ is also hyperbolic with distinct eigenvalues $\{\alpha_i, \lambda_j\}$ such that $|\alpha_i| < 1 < |\lambda_j|$. By hypothesis, we assume $|\lambda_1| < |\lambda_2|$. Now, choose a basis W_k of unit eigenvectors and renorm R^n by defining

$$\|\sum a_k W_k\|^2 = \sum |a_k|^2.$$

Let $\mu \in Z^n$ be a vector with nonzero projection onto the eigenvector corresponding to λ_2 and let $\hat{\eta}(\mu)$ be the eigenvector of unit length corresponding to λ_1. Define $\hat{\eta}(\nu) = B^k\hat{\eta}(\mu)$ if $\nu = B^k\mu$ and $\hat{\eta}(\nu) = 0$ otherwise. With this definition, $(A^{\#})^*\eta = \eta$. We complete the proof by showing $\eta \in H^{-1}(T^*M)$, i.e.

$$\sum_k \|\hat{\eta}(B^k\mu)\|^2(1 + \|B^k\mu\|^2)^{-1} < \infty.$$

From the choice of $\hat{\eta}(\mu)$, (1) gives

$$\|\hat{\eta}(B^k\mu)\|^2 = |\lambda_1|^{2k},$$

and for $k > 0$ the choice of norm gives

$$\|B^k\mu\|^2 \geqslant |\lambda_2|^{2k}|\mu_2|^2$$

where μ_2 is the projection of μ onto the eigenvector corresponding to λ_2. These estimates show that the summands are dominated by $|\lambda_1|^{2k}|\lambda_2|^{-2k}|\mu_2|^{-2}$ for $k > 0$ and by $|\lambda_1|^{2k}$ for $k < 0$. Hence, $\eta \in H^{-1}(T^*M)$. Q.E.D.

Example. The block diagonal matrix (A, B) given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

defines a hyperbolic toral automorphism on T^4 which is not infinitesimally ergodic.

Remark. In 2-dimensions, hyperbolic toral automorphisms *are* infinitesimally ergodic [**1**].

References

1. C. Chicone and R. Swanson, *Infinitesimal ergodicity for some dynamical systems* (preprint).

2. J. Robbin, *Topological conjugacy and structural stability for discrete dynamical systems*, Bull. Amer. Math. Soc. **78** (1972), 923–952.

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