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A NONSHRINKABLE DECOMPOSITION OF S^n DETERMINED BY A NULL SEQUENCE OF CELLULAR SETS

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ABSTRACT. The result derived is one of existence: there is a decomposition of the n-sphere S^n ($n > 5$) into points and a null sequence of cellular sets that is not shrinkable. In one form the implicit example leads to a nonmanifold decomposition space closely allied with the nonmanifold "dogbone" space developed by W. T. Eaton.

Among the roots of manifold decomposition theory is a fundamental example due to Bing [$\mathbf{B_1}$], usually called the dogbone space, the significance of which is its illumination of the negative aspects to the theory, standing as the first example of an upper semicontinuous decomposition of S^3 into points and cellular sets (namely, tame arcs) for which the associated decomposition space is *not* a topological manifold. After developing several conditions under which a decomposition of S^3 does yield S^3 as its decomposition space, Bing produced another example [$\mathbf{B_2}$] which did not; judged by almost every criterion the simplest possible such example, it is a decomposition of S^3 into points and a null sequence (that is, only finitely many elements of which have diameter greater than $\varepsilon > 0$) of cellular sets, each of which is contained in one of two fixed affine planes. Refinements to this phenomenom have been added by Gillman and Martin [**GM**] and by Bing and Starbird [**BS**], who showed in both cases that the null sequence could consist of cellular arcs (but not planar ones), and more recently by Starbird [**St**], who showed that the null sequence could consist of sets homeomorphic to any prescribed sequence of nondegenerate cellular subsets of S^3.

Extracting from past experience in which the pathology of embedding and/or decomposition theory manifested in 3-space is undiminished in higher dimensional manifolds, one expects comparable examples in S^n, $n \geqslant 4$. We shall establish the existence of such examples in S^n, $n \geqslant 5$, by applying to a large class of nonshrinkable decompositions an amalgamation technique employed by Edwards [**Ed**] in his recent characterization of shrinkable decompositions of S^n ($n \geqslant 5$). In the presence of a special hypothesis, the Disjoint Disk Property, Edwards showed that the shrinkability of a decomposition hinges on the shrinkability of certain closely related cellular

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decompositions, each determined by a null sequence of cellular elements and each element of restricted geometric embedding dimension; he also showed that all such restricted null sequence decompositions are shrinkable, Here, in the presence of a weaker hypothesis, the Disjoint Triples Property, we show that the shrinkability of a 0-dimensional decomposition of S^n hinges on the shrinkability of certain closely related decompositions determined by a null sequence of cellular elements. Since many nonshrinkable decompositions have the Disjoint Triples Property, there are many related nonshrinkable null sequence decompositions. In order to make the constructions nearly explicit, we shall take a particular nonshrinkable decomposition of S^n as a model from which we derive our examples by adjustment and amalgamation.

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We close this introduction by listing some conventions and definitions: all decompositions are upper semicontinuous decompositions into compact sets; given a decomposition G of a space X, we use π to denote the natural map of X to the associated decomposition space X/G; we say that π or any other map $f\colon X \to X/G$ is 1-1 over $A \subset X/G$ if $f|f^{-1}(A)$ is 1-1; in case X is a compact metric space, we say that G is *shrinkable* if for each $\varepsilon > 0$ there exists a homeomorphism h of X to itself such that the diameter of $h(g)$ is less than ε for every $g \in G$ and that

$$\operatorname{dist}(\pi(x), \pi h(x)) < \varepsilon \quad \text{for every } x \in X.$$

Reminder: If G is a decomposition of a closed n-manifold X into cell-like sets (see [**L**, §4]) and if $n \geqslant 5$ [**Si**], [**Ed**] or if $n = 3$ and the sets are cellular [$\mathbf{A_2}$], then G is shrinkable if and only if X/G is an n-manifold (necessarily homeomorphic to X).

1. A cellular null sequence. One possible model for our example is Eaton's nonmanifold dogbone space [**Ea**]. We find it advantageous to view this space as a sewing of a crumpled cube C to itself [**D**, Example 13.1], where C can be embedded in S^n so that $S^n - \operatorname{Int} C$ is an n-cell, locally flatly embedded except at points of a Cantor set tamely embedded in its boundary. In this form, the decomposition space Q can be expressed as the union of three pairwise disjoint sets, open sets U_1 and U_2, each homeomorphic to $\operatorname{Int} C$, and an $(n-1)$-sphere Σ. A significant feature of this representation is that Σ contains a Cantor set K, tamely embedded in Σ, such that $Q - K$ is an n-manifold. Generating this space Q is a cellular decomposition G of S^n, essentially determined by a parameterized annulus A, homeomorphic to $S^{n-1} \times I$, in such a way that the fiber arcs of A corresponding to $s \times I$, $s \in S^{n-1}$, are the nondegenerate elements of G, that Σ is the image of A in $Q = S^n/G$ under the decomposition map π, and that the closure of each component of $S^n - A$ is equivalent to C. Here $\operatorname{Bd} A$ is locally flat modulo $\operatorname{Bd} A \cap \pi^{-1}(K)$.

The first step of the plan is to identify a special F_σ set F in K such that

$Q - \Sigma$ is 1-ULC in $(Q - \Sigma) \cup F$ (each sufficiently small loop in $Q - \Sigma$ is contractible in a small subset of $(Q - \Sigma) \cup F$) and to produce a cell-like map f of S^n onto Q that is 1-1 over $F \cup (Q - K)$. This is described in the next section. The second half of the plan, completed in this section, is to amalgamate the nondegenerate inverse sets under f into a carefully controlled null sequence of cellular sets. With proper care, the associated null sequence decomposition cannot be shrinkable, for otherwise the original decomposition G would have been shrinkable as well, and Q would have been an n-manifold.

For now, we assume such a set F and map f have been found. Note that the decomposition G_f of S^n induced by f, where

$$G_f = \{f^{-1}(q) | q \in Q\},$$

has the nonmanifold Q as its decomposition space, which means that G_f is not shrinkable. In particular, there exists $\varepsilon > 0$ such that G_f does not satisfy the definition of shrinkability, for this positive number. At this juncture we want to exploit Edwards' amalgamation trick. Letting N_f denote the union of the nondegenerate point inverses of f ($N_f \subset f^{-1}(K - F)$), we see that N_f can be written as the union of compact sets $H_1, H_2, \ldots, H_k, \ldots,$ where each component of H_k is a point inverse of f having diameter $\geqslant 1/k$. The key is to cover $f(N_f)$ by a null sequence $\{Y_i\}$ of cell-like sets in $\Sigma - F$ such that each Y_i has diameter less than ε and that $\{f^{-1}(Y_i)\}$ is also a null sequence. It is easy enough to begin a construction with this aim: one covers $f(H_1) \subset K - F$ by a finite set of arcs in $\Sigma - F$, each having diameter less than $\varepsilon/2$. To continue, one expands these arcs, perhaps to dendrons, or perhaps so that the second stage minus the first is homeomorphic to [0, 1), but certainly to the same number of cell-like sets in $\Sigma - F$, so that $f(H_2)$ is covered by these plus finitely many arcs in $\Sigma - F$ having diameter less than $\varepsilon/4$. To continue further, one repeats such expansions, exercising controls to insure that each limiting set Y_i is a cell-like subset of $\Sigma - F$ (by arranging each as the closure of $\bigcup A_k$, where $A_1 \subset A_2 \subset \cdots$ and where A_k is contained in a neighborhood N_k such that each inclusion $N_k \to N_{k-1}$ is null homotopic and

$$\bigcap N_k = \text{closure}\Big(\bigcup A_k\Big)\Big),$$

and overseeing the construction so that each $f(H_k)$ is covered by finitely many of the Y_i's. The latter property implies that $\{f^{-1}(Y_i)\}$ is a null sequence in S^n.

Since f is cell-like, it is well known that each $f^{-1}(Y_i)$ is also cell-like [**L**, p. 511]. The crux, however, is that $f^{-1}(Y_i)$ is cellular. The place to measure cellularity is in Q: since $Y_i \subset Q - F$, Y_i satisfies McMillan's Cellularity Criterion (see [**M**]) in Q (by a simple Van Kampen argument, based on $Q - \Sigma$ being 1-ULC in $F \cup (Q - \Sigma)$). Therefore, $f^{-1}(Y_i)$, which satisfies the Cellularity Criterion in S^n [$\mathbf{A}_1$, Lemma 5.2], must also be cellular [**M**]. (This application of the Cellularity Criterion accounts for the restriction to $n \geqslant 5$;

it seems likely that with additional regulation one could determine cellularity in the case $n = 4$ as well.)

The decomposition G^* determined by the null sequence $\{f^{-1}(Y_i)\}$ is the one we want. It cannot be shrinkable, for otherwise G_f would have been. The main observation here is that each element of G_f is contained in some element of G^*, so any homeomorphism h of S^n to itself shrinking elements of G^* to small size necessarily does the same to elements of G_f. A minor technical point must be checked–that for π^*: $S^n \to S^n/G^*$ there is some $\delta > 0$ such that if h is a homeomorphism of S^n to itself for which $\operatorname{dist}(\pi^*(x), \pi^*h(x)) < \delta$ (in S^n/G^*), then $\operatorname{dist}(\pi(x), \pi h(x)) < \varepsilon$ (in Q)–but this is elementary uniform continuity.

2. A special F_σ set and a cell-like map. The required F_σ set F materializes from geometric properties of crumpled n-cubes C_i in S^n, where C_i denotes the closure of $\pi^{-1}(U_i)$, $i = 1, 2$. Any loop in Int C_i can be approximated by a PL simple closed curve in Int C_i which bounds a tame 2-cell D in S^n (with diameter approximately equal to that of the loop), and D meets Bd $C_i \cap \pi^{-1}(K)$ in a closed 0-dimensional set Z. Since Z is contained in $\pi^{-1}(K)$ and since $\pi|\text{Bd } C_i$ is a homeomorphism of Bd C_i to Σ, Z is tamely embedded in Bd C_i. If the original loop were small, it would be contractible in a fairly small subset of $Z \cup \text{Int } C_i$ (retract the part of D in $S^n - C_i$; approximate the resulting map, allowing changes only over the part in Bd C_i, to send it in $Z \cup (C_i - \pi^{-1}(K))$; and use the local flatness of Bd $C_i - \pi^{-1}(K)$ to push the new image into $Z \cup \text{Int } C_i$). Now list a dense subset of the loops in Int C_1 as $\{L_{2j+1}\}$ and a dense subset of the loops in C_2 as $\{L_{2j}\}$. Then for each loop L_{2j+1} (or L_{2j}) there corresponds a Cantor set Z_{2j+1} (or Z_{2j}) that is tame in S^n and such that L_{2j+1} (or L_{2j}) is contractible in a reasonably small subset of $Z_{2j+1} \cup \text{Int } C_1$ (or $Z_{2j} \cup \text{Int } C_2$). The desired F_σ set F is the union of the sets $\pi(Z_{2j+1})$ and $\pi(Z_{2j})$.

The map f will be the limit of cell-like maps $f_0 = \pi, f_1, f_2, \ldots$ (then f is necessarily cell-like [**L**, p. 505]) such that f_k is 1-1 over $\pi(Z_1 \cup \cdots \cup Z_k)$. In addition, f will be made 1-1 over $Q - K$ by expressing K as the intersection of open sets W_k and by requiring, not only that f_k be 1-1 over $Q - W_k$ and that f_{k+j} agree with f_k over $Q - W_k$, but also that the convergence of the sequence $\{f_{k+j} | j = 1, 2, \ldots\}$ submit to controls guaranteeing

$$f|f^{-1}(\pi(Z_k) \cup (Q - W_k)) = f_k|f_k^{-1}(\pi(Z_k) \cup (Q - W_k)).$$

To begin the construction, over $Q - W_1$ $f_0 = \pi$ fails to be 1-1 on $\pi^{-1}(\Sigma - W_1)$, which is contained in a locally flat part of the annulus A. Thus, there is an obvious map θ of S^n to itself shrinking each fiber arc $\pi^{-1}(x)$, $x \in \Sigma - W_1$, to a point of itself; furthermore, but less obviously, this map can be arranged so as to shrink each arc $\pi^{-1}(z)$, $z \in \pi(Z_1)$, to the point $\pi^{-1}(z) \cap$ Bd C_2, in such a way that θ has no inverse sets other than those described above. The latter shrinking follows because there are finitely many embeddings ψ of $B^{n-1} \times I$ into $A \cap \pi^{-1}(W_1)$ such that the images of Int $B^{n-1} \times I$

cover $\pi^{-1}(\pi(Z_1))$, each fiber $\pi^{-1}(z)$ corresponds to some ψ (point $\times$ I), and $\psi(B^{n-1} \times 1)$ is locally flat modulo Z_1. Since Z_1 is twice flat, flat both in S^n and in $\psi(B^{n-1} \times 1)$, $\psi(B^{n-1} \times 1)$ is flat [**K**, Theorem 1], and the appropriate shrinking can then occur near these cells $\psi(B^{n-1} \times I)$. Define $f_1 = f_0\theta^{-1}$ and note that $f_1^{-1} = \theta f_0^{-1}$, which is 1-1 over $\pi(Z_1) \cup (Q - W_1)$.

To continue, one must note that $f_1^{-1}(\Sigma) = \theta f_0^{-1}(\Sigma) = \theta(A)$, and then one must perform similar operations from the other side $\theta(C_2)$ on some fibers in the pinched annulus $\theta(A)$, shrinking out arcs in $\theta\pi^{-1}(\Sigma - W_2)$ and shrinking out the partially pinched family of arcs in $\theta\pi^{-1}(\pi(Z_2))$ toward the $\theta(\mathrm{Bd}\, C_1)$ side. This forms the typical iterative step, and successive steps alternate in directions, first moving from the moved C_1 to the moved C_2, and then in the reverse direction. After f_k has been constructed, and given $\varepsilon_k > 0$, one can demand that

$$\mathrm{dist}(f_k(s), f_{k+1}(s)) < \varepsilon_k \quad \text{for each } s \in S^n$$

to guarantee that the sequence $\{f_k\}$ converges to a map f.

3. Generalizations. To what extent do the constructions of the previous sections depend upon the model dogbone space Q? For those who may seek nonshrinkable decompositions of this sort based upon different nonmanifold spaces, we set forth some properties under which similar procedures will work.

THEOREM. *Suppose that G is a cellular decomposition of a closed n-manifold M, $n \geqslant 5$, such that $M/G = Q$ is a nonmanifold and the image under the natural map π: $M \to M/G = Q$ of the nondegenerate elements is 0-dimensional. Suppose also that Q has the following Disjoint Triples Property: any three maps m_i $(i = 1, 2, 3)$ of B^2 into Q can be approximated arbitrarily closely by maps m_i' of B^2 into Q such that*

$$m_1'(B^2) \cap m_2'(B^2) \cap m_3'(B^2) = \varnothing.$$

Then there exists a cell-like map f: $M \to Q$ approximating π and there exist cell-like sets $\{Y_i\}$ in Q such that $\{f^{-1}(Y_i)\}$ forms a null sequence of cellular sets generating a nonshrinkable decomposition of M.

The techniques for verifying this differ somewhat from those used here. The Disjoint Triples Property translates into the existence of a 2-dimensional F_σ set F in Q such that any cell-like set in $Q - F$ satisfies the Cellularity Criterion in Q and, for each compact subset T of F, the induced decomposition

$$G_T = \left(M - \pi^{-1}(T)\right) \cup \left\{\pi^{-1}(t) | t \in T\right\}$$

is shrinkable (satisfies the Disjoint Disk Property). Here F can be described with the aid of a sequence of embeddings f_i of $B_1^2 \cup B_2^2 \cup B_3^2$ (disjoint 2-cells) in M such that $\{\pi \circ f_i\}$ is dense in the space of maps into Q and such that

$$\pi f_i(B_1^2) \cap \pi f_i(B_2^2) \cap \pi f_i(B_3^2) = \varnothing$$

for each i, by defining F as $\bigcup \pi f_i(B_1^2)$. Edwards' main result **[Ed]**, applied repeatedly, permits approximation of π by a map f that is 1-1 over F and that fails to be 1-1 only over (some) points of a 0-dimensional subset of Q (namely, a G_δ set containing the original 0-dimensional set in Q). Since Q is a generalized n-manifold and satisfies duality, (see **[An]**), the 2-dimensional set F neither contains nor separates any open subset of Q. As a result, one can build a null sequence $\{Y_i\}$ of cell-like sets in $Q - F$ containing the nondegenerate inverse sets of f, as before.

Dennis Garity has informed me that a converse to this theorem is valid, which implies then that such a decomposition space Q has the Disjoint Triples Property if and only if there exists a cell-like map f: $M \to Q$ satisfying the conclusions of the Theorem.

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