EMBEDDING PHENOMENA BASED UPON DECOMPOSITION THEORY: WILD CANTOR SETS SATISFYING STRONG HOMOGENEITY PROPERTIES

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Abstract. We point out the sharpness of earlier results of McMillan by exhibiting a map of the n-sphere $S^n$, $n > 5$, onto itself having acyclic but non-cell-like polyhedra as its nondegenerate point inverses and for which the image of the set of nondegenerate point inverses is a Cantor set $K$. Of necessity, $K$ is wildly embedded, and it has the unusual additional property that every self-homeomorphism of $K$ extends to a self-homeomorphism of $S^n$.

1. Introduction. According to work of D. R. McMillan, if $f$ is a map of $S^n$ to itself such that the image of the set of nondegenerate point inverses is 0-dimensional, then each point inverse is strongly acyclic over the integers (see [M] for definitions) and, in particular, has the integral Čech cohomology of a point [M, Lemma 5]; moreover, for the case $n = 3$, each point inverse is cellular [M, Corollary 3.5]. We show here that for $n > 5$ this stronger conclusion of cellularity fails in what is known to be the simplest possible case, in which the image of the nondegenerate elements forms a Cantor set.

This image Cantor set $K$ must be wildly embedded (otherwise, $K$ would be defined by cells in $S^n$, and the inverse image of the defining cells would also be cells, implying that each nondegenerate point inverse is cellular). As an elementary by-product of its construction, $K$ is seen to possess a symmetry previously undiscovered in wild Cantor sets, for it is strongly homogeneously embedded, meaning that each homeomorphism of $K$ onto itself can be extended to a homeomorphism of $S^n$ to itself. Displaying a weaker form of symmetry, the classical examples of Antoine [A] and Blankenship [Bl] are homogeneously embedded in the sense that, for any two points $p$, $q$ in such examples $X$, there is a homeomorphism $H$ of $S^n$ to itself for which $H(X) = X$ and $H(p) = q$.

Some profound recent developments concerning decompositions of manifolds support what may appear to be the innocuously easy constructions of this paper. The first of these is due to Cannon [C], who showed that for a
cell-like decomposition $G$ of an $n$-manifold $M$ ($n > 5$) such that the image of the nondegenerate elements is contained in a closed $k$-dimensional set $Y$ ($2k + 1 < n$), $M/G$ is a manifold (homeomorphic to $M$) if and only if $M/G$ satisfies the following disjoint disks property: any two maps of the 2-cell $I^2$ into $M/G$ can be approximated arbitrarily closely by maps having disjoint images. This has been improved by Edwards [E], who obtained the same result with no restriction on the image of the nondegenerate elements beyond the requirement that $M/G$ itself be finite dimensional. The second development, concerning the resolution of singularities in nonmanifolds, is due to Bryant and Lacher [BL], who showed that if $Y$ is a generalized $n$-manifold of dimension $n > 5$ that is known to be an $n$-manifold except possibly at points of some 0-dimensional closed subset $S(Y)$, then $Y$ is the cell-like image of an $n$-manifold. This result also has been improved, by Cannon, Bryant and Lacher [CBL], who obtained the same conclusion in case the potential nonmanifold set $S(Y)$ is contained in a closed subset of dimension $k$, where $2k + 2 < n$. An explicit consequence of the above needed for applications here is the following theorem.

**Theorem A (Cannon, Bryant and Lacher).** Suppose $Y$ is a generalized $n$-manifold, $n > 5$, such that

1. $Y$ contains a 0-dimensional closed set $S(Y)$ such that $Y - S(Y)$ is an $n$-manifold, and
2. $Y$ satisfies the disjoint disks property.

Then $Y$ is an $n$-manifold.

As in [BL] and [CBL], a generalized $n$-manifold is understood to be an ENR (Euclidean neighborhood retract = a retract of an open subset of some Euclidean space) such that, for each $y \in Y$,

$$H_*(Y, Y - \{y\}; Z) \approx H_*(E^n, E^n - \{0\}; Z).$$

2. **The basic construction.** McMillan [M, p. 959] presents an example of an acyclic but non-cell-like map $f$ of $S^n$ ($n > 4$) to itself such that the image of the nondegenerate elements is an arc. To a great extent the example described below represents a 0-dimensional version of his.

Throughout the remainder of this paper $n$ will represent a fixed integer greater than 4.

Let $M^{n-2}$ be a compact PL homology $(n-2)$-cell (an $(n-2)$-manifold with-boundary having trivial homology groups but nontrivial fundamental group) and let $X'$ be a PL $(n-3)$-spine for $M^{n-2}$, that is, $X' \subset \text{Int } M^{n-2}$ and $M^{n-2}$ collapses to $X'$. Let $N^{n-1} = M^{n-2} \times [-1, 1]$, which then has $X = X' \times \{0\}$ as a spine and for which, in particular, $N^{n-1} - X \approx (\partial N^{n-1}) \times [0, 1]$. Let $C$ be the standard “middle thirds” Cantor set in $I = [0, 1]$. Consider the upper semicontinuous decomposition $G$ of $Q = N^{n-1} \times [-2, 2]$ having $\{X \times \{c\} | c \in C\}$ as its collection of nondegenerate elements. Let $Q^*$
denote the decomposition space \( Q/G \) and \( \pi: Q \rightarrow Q^* \) the decomposition map.

**Main Lemma.** The decomposition space \( Q^* = Q/G \) is a compact \( n \)-manifold-with boundary.

**Proof.** Clearly the image of \( \partial Q \) in \( Q^* \) is a collared \( (n-1) \)-manifold. The argument here will establish that the image \( Y \) of \( \text{Int } Q \) is an \( n \)-manifold.

The space \( Y \) contains a Cantor set \( K \) of possible singular points, \( K \) corresponding to the image under \( \pi \) of the nondegenerate elements of \( G \), such that \( Y - K \) is an \( n \)-manifold. Not only does this mean that \( Y \) fulfills condition (1) of Theorem A, it also implies that \( Y \) is \( n \)-dimensional [HW, p. 32].

Next we show that \( Y \) is locally contractible. This is obvious for points of \( Y - K \). Since each point of \( K \) has arbitrarily small (closed) neighborhoods homeomorphic to \( Q^* \), it suffices to prove that \( Q^* \) is contractible. The construction guarantees that \( Q^* \) deformation retracts to \( \pi(X \times I) \), and thus the problem reduces further to proving that \( \pi(X \times I) \) is contractible. To do that, we name two auxiliary sets of maps. The first is a set of retractions \( r_c \), defined for \( c \in C \), of \( \pi(X \times I) \) to \( \pi(X \times [c, 1]) \) sending \( \pi(X \times [0, c]) \) to the point \( \pi(X \times \{c\}) \). Before we name the second, we note that for each component \((a, b)\) of \( I - C \), \( \pi(X \times [a, b]) \) is topologically the suspension of the acyclic polyhedron \( X \) and, therefore, is contractible (see [S, Exercise 8.D.3, p. 461]). Then the second auxiliary set is a family of contractions, where, for each component \((a, b)\) of \( I - C \), \( \psi_t \) is a contraction of \( \pi(X \times [a, b]) \), parametrized by \( t \in [a, b] \), such that \( \psi_0 \) is the identity, \( \psi_b(\pi(X \times \{b\})) \) is identically \( \pi(X \times \{b\}) \), and \( \psi_b \) is the constant map to \( \pi(X \times \{b\}) \). Now we define a contradiction \( h_t (t \in I) \) of \( \pi(X \times I) \) as

\[
h_t(\pi(x, s)) = \begin{cases} 
\pi(x, s) & \text{if } s > t, \\
r_t(\pi(x, s)) & \text{if } t \in C, \\
\psi_t r_x(x, s) & \text{otherwise}
\end{cases}
\]

where the convention (• • •) governing the final part of this rule is that \( s < t \) and \( t \) lies in the component \((a, b)\) of \( I - C \).

It follows that the finite dimensional, locally contractible separable metric space \( Y \) is an ANR [H, Theorem V.7.1] and, therefore, is an ENR [L, p. 718]. Moreover, because each \( \pi^{-1}(y) \) is acyclic, the Vietoris-Begle mapping theorem [Br, Theorem V.6.1] and standard duality theory [S, Theorem 6.9.10] yield that

\[
H_{n-k}(Y, Y - \{y\}) \approx H_{n-k}(\text{Int } Q, \text{Int } Q - \pi^{-1}(y)) \\
\approx \overline{H}^k(\pi^{-1}(y)) \\
\approx H^k(\text{point}) \\
\approx H_{n-k}(E^n, E^n - \{0\}).
\]
As a result, $Y$ is a generalized $n$-manifold.

Finally, we turn to condition (2) of Theorem A—the disjoint disks property. We first show that, for any dense subset $D$ of $K$, each map $f$ of $I^2$ into $Y$ can be approximated by a map of $I^2$ into $D \cup (Y - K)$. To do this, choose a triangulation $T$ of $I^2$ with very small mesh. Approximate $f$ by a map $g$ such that $g(T^{(1)}) \subset Y - \pi(X \times [-2, 2])$ (here $T^{(1)}$ denotes the 1-skeleton of $T$), which is possible, of course, because $\dim \pi(X \times [-2, 2]) < n - 2$. Require this approximation $g$ to be so close to $f$ that, for those 2-simplexes $\sigma$ of $T$ such that $f(\sigma)$ misses $K$, $g(\sigma)$ also misses $K$. In case $f(\sigma) \cap K \neq \emptyset$, modify $g|\sigma$ once more in the following manner: $g|\sigma$ is homotopic to a small loop $L$ near the cone point $\pi(X \times \{d\})$ in the space $\pi(N^{n-1} \times \{d\})$, $d \in D$ (which space is topologically the cone on $\partial N^{n-1}$), by a homotopy moving points along the images of vertical arcs from $Q = N^{n-1} \times [-2, 2]$ and ranging through a small subset of $Y - \pi(X \times [-2, 2])$; the loop $L$ then is contractible in a small subset of $\pi(N^{n-1} \times \{d\})$. Define $g|\sigma$ as such a contraction of $g|\sigma$.

In order to establish the disjoint disks property, we choose disjoint, dense subsets $D_1$ and $D_2$ of $K$. By the preceding paragraph, given maps $f_i$ of $I^2$ into $Y$ ($i = 1, 2$), we can approximate them by maps $g_i$ such that $g_i(I^2) \subset D_i \cup (Y - K)$ ($i = 1, 2$). This means that $g_1(I^2)$ and $g_2(I^2)$ intersect only at points of the $n$-manifold $Y - K$. Consequently, we can exploit traditional general position methods to further adjust the maps $g_i$, changing things only at points of $g_i^{-1}(Y - K)$, to maps $h_i$ ($i = 1, 2$) such that

$$h_1(I^2) \cap h_2(I^2) = \emptyset.$$  

As a consequence of Theorem A, $Q^*$ is an $n$-manifold-with boundary.

3. The map of $S^n$ to itself.

**Proposition 1.** There is a non-cell-like map $f$ of $S^n$ to itself ($n \geq 5$) such that the image of the nondegenerate point inverses under $f$ is a Cantor set $K$.

**Proof.** Crucial to this argument is a fact established in the course of the main lemma that $Q^*$ is contractible.

Form a space $S$ from the disjoint union of $Q$ and $Q^*$ by identifying each point $x \in \partial Q$ with $\pi(x) \in Q^*$, form another space $T$ by doubling $Q^*$ along $\partial Q^*$ ($T$ results from the disjoint union of two copies of $Q^*$ by identifying corresponding boundary points), and name a map $f$ of $S$ onto $T$ such that $f|Q$ acts like $\pi$ in taking $Q$ onto one of the copies of $Q^*$ and that $f|Q^*$ acts as the identity mapping onto the other copy of $Q^*$ in $T$. Then the set of nondegenerate point inverses of $f$ coincides with that of $\pi$, and its image under $f$ is a Cantor set $K$ in $T$.

Each of $S$ and $T$ is a closed $n$-manifold. By a simple Mayer-Vietoris calculation, each has the homology of $S^n$. Moreover, each is simply connected: since $Q^*$ is contractible, $\pi_1(S)$ is generated by the image of $\pi_1(Q)$, which in turn is generated by the image of $\pi_1(\partial Q)$, and which itself is contained in
the (trivial) image of \( \pi_1(Q^*) \); it is even more obvious that \( T \) is simply connected. Hence, \( S \) and \( T \) are each topologically equivalent to \( S^n[N] \).


**Proposition 2.** There exists a wildly embedded, strongly homogenously embedded Cantor set \( K \) in \( S^n \) \( (n > 5) \). Furthermore, for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that every homeomorphism \( h \) of \( K \) to itself moving points less than \( \delta \) extends to a homeomorphism \( H \) of \( S^n \) to itself moving points less than \( \varepsilon \) and fixed outside the \( \varepsilon \)-neighborhood of \( K \).

**Proof.** The Cantor set \( K \), of course, is the one determined in §3, where \( K \subset Q^* \subset T \approx S^n \). As an alternative to the decomposition theory argument sketched in the introduction that \( K \) is wild, consider a map \( g: \partial I^2 \to N^{n-1} \times \{2\} \) defining a loop in \( \partial Q \) that is not contractible in \( Q \). Since \( Q^* \) is contractible, \( \sigma g \) extends to a map \( g^* \) of \( I^2 \) in \( Q^* \). If \( K \) were tame, \( g^* \) could be adjusted, without changing the map on \( \partial I^2 \), to a map \( g' \) into \( Q^* - K \). This leads to the contradiction that \( \pi^{-1}g' \) is a contraction of \( g \) in \( Q \).

As an aid for studying the embedding of \( K \) in \( Q^* \), we reconsider the source \( Q \) as \( M^{n-2} \times B \), with \( B \) representing the 2-cell \([-1,1] \times [-2,2] \), and with \( C = \{0\} \times C \) the Cantor set in \( \text{Int } B \) for which \( \pi(X' \times C) = K \). The tameness of Cantor sets in the plane implies that each homeomorphism \( h^* \) of \( C \) onto itself extends to a homeomorphism \( H^* \) of \( B \) onto itself fixed on \( \partial B \). Then, given any homeomorphism \( h \) of \( K \) to itself, one induces a homeomorphism \( h^* = \pi^{-1}h\pi \) on \( C \), extends \( h^* \) to the promised homeomorphism \( H^* \) on \( B \), defines a homeomorphism \( H \) on \( Q^* = \pi(M^{n-2} \times B) \) as \( \pi(\text{Id} \times H^*)\pi^{-1} \), and finally extends \( H \) to other points of \( T \approx S^n \) via the identity. Furthermore, because \( C \subset B^2 \) satisfies the stronger homogeneity property mentioned in the statement of the proposition, the argument just given shows that \( K \subset S^n \) satisfies it as well.

**References**


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