EMBEDDING PHENOMENA BASED UPON DECOMPOSITION
THEORY: AN UNUSUAL CELLULAR 2-CELL

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Abstract. We construct a 2-cell D cellularly embedded in the n-sphere S^n, n > 5, and locally flatly embedded at each of its interior points, but the boundary of which fails to be weakly flat.

1. Introduction. A k-sphere in S^n is said to be weakly flat if its complement is homeomorphic to the complement in S^n of the standardly embedded k-sphere. The origin of this concept is found in work of McMillan [9], who showed that an (n − 1)-sphere S in S^n, n > 5, is weakly flat if and only if it satisfies the following cellularity criterion: For each open set U containing S there exists an open set V containing S such that each loop in V − S is contractible in U − S. Duvall [4] introduced the “weak flatness” terminology and proved that a k-sphere S in S^n, 2 ≤ k ≤ n − 3, is weakly flat if and only if it satisfies McMillan’s cellularity criterion. The two cases k = 1 and k = n − 2 demand a different homotopy condition: a closed subset X of S^n is said to be globally 1-alg if for each open set U containing X there exists an open set V containing X such that each loop in V − X that is null homologous in V − X is also null homotopic in U − X. This new condition is a reasonable variation to the cellularity criterion, for in case X is a k-sphere (k ≠ 1, n − 2) or a cell-like subset of S^n, one can easily show that X is globally 1-alg if and only if X satisfies the cellularity criterion. Completing the characterization of weak flatness for spheres, Daverman [2] showed that a 1-sphere in S^n, n > 5, is weakly flat if and only if it is globally 1-alg, and Hollingsworth and Rushing [7] did the same for (n − 2)-spheres S in S^n, n > 5, under the necessary additional hypothesis that S^n − S have the homotopy type of S^1.

There is one long-standing puzzle concerning weak flatness. Duvall proved that the boundary of each cellular k-cell, 3 ≤ k ≤ n − 2, in S^n is weakly flat, and he raised the related question about the boundary of a cellular 2-cell [4]. Previously McMillan’s work [9] had given an affirmative answer for the case of a cellular n-cell (n > 5), and almost immediately after weakly flat (n − 2)-spheres were characterized, Daverman and Rushing [3] gave an affirmative
answer for the case of a cellular \((n - 1)\)-cell. This note answers Duvall’s question negatively.

The heart of this paper is the Cannon-Edwards [1], [5], [6] solution to the double suspension problem, for purposes here best expressed in the following form: If \(S\) is a closed \((n - 2)\)-manifold with the homology groups of \(S^{n-2}\), then the product of \(E^1\) with the cone over \(S\) is an \(n\)-manifold. There is sufficient freedom in the construction that, if one prefers, one can consider only those homology spheres \(S\) that do bound contractible manifolds, thereby eliminating the difficult problem of realizing this product space as a cell-like decomposition of a manifold.

2. The example. First we outline the steps of the construction. After completing the outline, we supply further details for those steps marked with an asterisk. Throughout this section \(n\) denotes a fixed integer greater than 4.

Step 1*. Find a PL homology \((n - 2)\)-sphere \(S\) with non trivial fundamental group \(\pi\) that can be trivialized by adding relations given by some inner automorphism \(\psi\) of \(\pi\) (more precisely, the normal closure of \(\{g^{-1}\psi(g)\mid g \in \pi\}\) is \(\pi\) itself).

Step 2*. Determine a PL homeomorphism \(f\) of \(S\) to itself leaving a neighborhood of some \(p \in S\) pointwise fixed and for which the induced homeomorphism on fundamental groups is \(\psi\).

Step 3. Form the cone \(cS\) on \(S\), produce a homeomorphism \(F\) of \(cS\) to itself by coning over \(f\), and construct the mapping torus \(T\) of \(F\). Explicitly, \(T\) issues from \((cS) \times [0, 1]\) by identifying each point \((x, 0)\) with \((F(x), 1)\). According to the double suspension theorem of Cannon-Edwards [1], [5], [6], \(T\) is a compact \(n\)-manifold-with-boundary.

Step 4. Observe that the simple closed curve \(J\) in \(\partial T\) corresponding to \(\{p\} \times [0, 1]\) has a tubular neighborhood homeomorphic to \(B^{n-1} \times S^1\). Add a 2-handle \(H\) to \(T\) along this neighborhood, with \(J\) as the attaching sphere, to determine a new \(n\)-manifold-with-boundary \(M\).

Step 5*. Specify a 2-cell \(D\) in \(\text{Int} M\) as the core of the 2-handle \(H\) plus the annulus corresponding to \((\text{cone over } p) \times [0, 1]\), after identification, in \(T\). The first claim is that \(D \subset M\) satisfies the cellularity criterion, and, therefore, has a neighborhood embeddable in \(S^n\). The second claim is that the image of \(\text{Bd} D\) under such an embedding cannot be weakly flat.

Now for some details. In case \(n = 5\), a perfectly good homology 3-sphere \(S\) is the Mazur example [8], the fundamental group \(\pi\) of which has the presentation

\[
\{ x, y | y^4 = (xy)^2y(xy)^2, (xy)^2 = y(xy)^2x^{-1}(y^{-1}x^{-1})^2 \},
\]

but any other example with 2-generator group would work equally well. For the inner automorphism \(\psi\) on \(\pi\) induced by one of these generators, say \(y\), the normal closure \(N\) of the element \(x^{-1}\psi(x) = x^{-1}yx\) equals \(\pi\), since \(\pi/N\), being abelian, must be trivial. In case \(n > 5\), a homology \((n - 2)\)-sphere \(S'\) with fundamental group \(\pi\), as above, can be given by letting \(B\) denote the
complement of the interior of a PL 3-cell in $S$ and by defining $S'$ as the double of $B \times I^{n-5}$ (that is, $S'$ equals the closed manifold obtained from two copies of $B \times I^{n-5}$ by identifying corresponding boundary points).

To determine the appropriate homeomorphism $f$ on $S$, simply transport the base point $s$ of $\pi_1(S, s)$ around the element giving rise to $\psi$ while keeping a large open set pointwise fixed.

Before showing that $D$ satisfies the cellularity criterion, we prove that $\partial M$ is simply connected. Clearly $\pi_1(dM)$ results from $\pi_1(D)$ by killing the group element determined by $J$. The group $\pi_1(\partial T)$ is the semidirect product of $\pi_1(S)$ and the integers $\mathbb{Z}$, represented by adding an extra generator $v$ (corresponding to $J$) to $\pi_1(S)$, subject to the additional relations $v \cdot \alpha \cdot v^{-1} = f_\ast(\alpha)$ for each $\alpha \in \pi_1(S)$. Killing $v$ reduces the computation to adding all relations $\alpha = f_\ast(\alpha)$ for each $\alpha \in \pi_1(S)$ to $\pi_1(S)$, and by construction of $\psi$ and $f_\ast$, this results in the trivial group.

The argument above is the key to the geometry. With it we can readily establish the two principal claims.

**Claim 1.** $D$ satisfies the cellularity criterion in $M$.

It is a direct consequence of the description of $M$ as a simplicial (but not PL!) regular neighborhood of $D$ that $D$ has arbitrarily small neighborhoods $M'$ simplicially isomorphic to $M$, for which then $M' - D \cong \partial M \times [0, 1)$ is simply connected, implying that $D$ satisfies the cellularity criterion. One might note that doubling $M$ produces a homotopy $n$-sphere, necessarily $S^n$, and quickly embeds $M$ itself in $S^n$.

**Claim 2.** For $\text{Bd} \ D \subset M \subset S^n$, $\text{Bd} \ D$ fails to be weakly flat.

Here $T \subset M \subset S^n$. Then $\text{Bd} \ D$ is contained in $T$ in such a way that the infinite cyclic cover $\tilde{T}'$ of $T$ is the space $(cS) \times E^1$, in which the set $X$ covering $\text{Bd} \ D$ is the line given by the product of the cone point with $E^1$. Then $T' - X \approx S \times [0, 1) \times E^1$, and $X$ fails to be globally 1-alg in $T'$. It follows easily that $\text{Bd} \ D$ cannot be globally 1-alg in $T$, so $\text{Bd} \ D \subset S^n$ cannot be weakly flat [2].

**References**


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