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A JORDAN FACTORIZATION THEOREM FOR POLYNOMIAL MATRICES

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ABSTRACT. It is shown that a complex polynomial matrix $M(\lambda)$ which has a proper rational inverse can be factored into $M(\lambda) = \hat{C}(\lambda)(\lambda I - J)\hat{B}(\lambda)$ where J is a matrix in Jordan normal form and the columns of $\hat{C}(\lambda)$ consist of eigenvectors and generalized eigenvectors of a linear operator associated with $M(\lambda)$. For a proper rational matrix W with factorizations $W(\lambda) = C(\lambda I - J)^{-1}B = M(\lambda)^{-1}P(\lambda) = Q(\lambda)N(\lambda)^{-1}$ it will be proved that C consists of Jordan chains of M and B of Jordan chains of N.

1. Introduction. Any linear polynomial matrix $L(\lambda) = \lambda I - A$, $A \in \mathbf{C}^{n\times n}$, admits two distinct canonical factorizations. One is given by

$$L(\lambda) = F(\lambda)D(\lambda)G(\lambda) \tag{1.1}$$

where F and G are unimodular and $D = \operatorname{diag}(d_1, \ldots, d_n)$ is the Smith form of L. The other is

$$L(\lambda) = C(\lambda I - J)B \tag{1.2}$$

where J is the Jordan form of A, $B = C^{-1}$, and the columns of C are chains of eigenvectors and generalized eigenvectors of A. In the case of a general complex polynomial matrix $M \in \mathbf{C}^{n\times n}[\lambda]$,

$$M(\lambda) = \sum_{\mu=0}^{m} A_\mu \lambda^\mu, \qquad A_\mu \in \mathbf{C}^{n\times n},$$

only a factorization of type (1.1) has been considered. It is the purpose of this note to show that M too can be factored into a product of type (1.2),

$$M(\lambda) = \hat{C}(\lambda)(\lambda I - J)\hat{B}(\lambda),$$

provided M has a proper rational inverse. The columns of $\hat{C}(\lambda)$ consist of eigenvectors and generalized eigenvectors of a linear operator associated with M. Our investigation uses Fuhrmann's approach [4] to the realization of rational matrices.

2. Jordan chains of polynomial matrices. We shall use the following notation. $\mathbf{C}^n[\lambda]$ will denote the space of all column vectors with complex polynomial entries. Similarly $\mathbf{C}^{n\times k}[\lambda]$ denotes the vector space of all $n \times k$ complex polynomial matrices. The entries of vectors in $\mathbf{C}^n(\lambda)$ and of matrices in $\mathbf{C}^{n\times k}(\lambda)$ are complex rational functions. A complex function f is called

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proper rational, if it is the quotient of two polynomials, $f = p/q$, and $\deg p < \deg q$. An element of $\mathbf{C}^n(\lambda)$ or $\mathbf{C}^{n\times k}(\lambda)$ is proper rational if all of its entries are proper rational.

For the definitions and results of this section we refer to [1], [6], [7]. A number $\lambda_0 \in \mathbf{C}$ is called a *characteristic root* of $M \in \mathbf{C}^{n\times n}[\lambda]$, if $\det M(\lambda_0) = 0$. A *Jordan chain* of M corresponding to the characteristic root λ_0 is a sequence of vectors $c_1, c_2, \ldots, c_s, c_\sigma \in \mathbf{C}^n$, which satisfy $c_1 \neq 0$ and

$$\sum_{p=0}^{j} \frac{1}{p!} M^{(p)}(\lambda_0) c_{j+1-p} = 0, \qquad j = 0, 1, \ldots, s-1. \tag{2.1}$$

The chain $c_1, \ldots, c_s$ of length s will be called *full*, if it cannot be extended to a chain of length $s+1$. The sequence $c_s, c_{s-1}, \ldots, c_1$ is called a *reversed chain*. In the special case of $M(\lambda) = \lambda I - A$, $A \in \mathbf{C}^{n\times n}$, the relations (2.1) yield

$$(\lambda_0 I - A)c_1 = 0, \quad (\lambda_0 I - A)c_{i+1} = c_i, \qquad i = 1, \ldots, s-1,$$

and a Jordan chain of M consists of an eigenvector c_1 and generalized eigenvectors of A.

We shall use J for a matrix in (upper triangular) Jordan form,

$$J = \text{block diag}(J_1, \ldots, J_g), \tag{2.2}$$

where

$$J_\gamma = \begin{bmatrix} \lambda_\gamma & 1 & 0 & \cdots & 0 \\ 0 & \lambda_\gamma & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdot & \cdots & \lambda_\gamma \end{bmatrix}.$$

The columns of a matrix $C \in \mathbf{C}^{n\times r}$ are said to form a *full system* [7] of Jordan chains of $M(\lambda) = \sum_{\mu=0}^{m} A_\mu \lambda^\mu$, if

$$r = \deg \det M \tag{2.3}$$

and if there is a matrix $J \in \mathbf{C}^{r\times r}$ such that

$$\sum_{\mu=0}^{m} A_\mu C J^\mu = 0 \tag{2.4}$$

and

$$\operatorname{rank} \begin{bmatrix} C \\ CJ \\ \vdots \\ CJ^{r-1} \end{bmatrix} = r. \tag{2.5}$$

The conditions (2.3)–(2.5) are equivalent (see [7]) to the property that the columns of Ce^{Jt} form a fundamental system of solutions of the differential

equation

$$\sum_{\mu=0}^{m} A_\mu x^{(\mu)}(t) = 0.$$

If J has the block structure (2.2), then C can be partitioned into $C = (C_1, \ldots, C_g)$ such that the columns of each C_γ form a full chain. Analogously the columns of B^T are said to form a *full system of reversed chains* of M^T if $\sum J^\mu B A_\mu = 0$ and

$$\operatorname{rank}(B, JB, \ldots, J^{r-1}B) = r. \tag{2.6}$$

In the case $M(\lambda) = \lambda I - A$ the preceding definitions yield nonsingular matrices C and B such that

$$\lambda I - A = C(\lambda I - J)C^{-1} = B^{-1}(\lambda I - J)B.$$

3. Realizations. Let $W \in \mathbf{C}^{k\times n}(\lambda)$ be a proper rational matrix. Thus W has a formal expansion

$$W(\lambda) = \sum_{\mu=0}^{\infty} W_i \lambda^{-(i+1)}.$$

LEMMA 1 [4]. *There is an invertible matrix $M \in \mathbf{C}^{k\times k}[\lambda]$ and a matrix $P \in \mathbf{C}^{k\times n}[\lambda]$ such that*

$$W = M^{-1}P \tag{3.1}$$

and M and P are left coprime. M and P are unique up to a common left unimodular factor. Analogously there exist an invertible matrix $N \in \mathbf{C}^{n\times n}[\lambda]$ and a matrix $Q \in \mathbf{C}^{k\times n}[\lambda]$ such that

$$W = QN^{-1} \tag{3.2}$$

and N and Q are right coprime.

DEFINITION (see e.g. [2], [4]). A triple of linear operators $[A, B, C]$, B: $\mathbf{C}^n \to X$, A: $X \to X$, C: $X \to \mathbf{C}^k$ is called a *realization* of W, if

$$W_i = CA^iB, \qquad i = 0, 1, 2, \ldots.$$

The realization is called *minimal*, if the dimension of the state space X is minimal.

We note the following result on realizations of W.

LEMMA 2 (see [3]). *If p is the least common denominator of all minors of W and $r = \deg p$, then there is a minimal realization $[J, B, C]$ of W with state space $X = \mathbf{C}^r$ such that*

$$W(\lambda) = C(\lambda I - J)^{-1}B \tag{3.3}$$

and $J \in \mathbf{C}^{r\times r}$ is in Jordan form. For any minimal realization J, B, C the conditions (2.5) *and* (2.6) *hold and the characteristic polynomial of J is equal to p.*

Since (2.5) is equivalent to $\lambda I - J$ and C being right coprime and (2.6) to

$\lambda I - J$ and B being left coprime, the last statement of Lemma 2 is contained in the following result.

LEMMA 3 [3, p. 102]. *If P, Q and S are polynomial matrices such that S is left coprime to P and right coprime to Q, then the least common denominator of all minors of $W = QS^{-1}P$ is equal to* det S.

The matrices C and B in (3.3) can be characterized in terms of Jordan chains of M and N in (3.1) and (3.2).

THEOREM 1. *If W has the representations*

$$W(\lambda) = C(\lambda I - J)^{-1}B = M^{-1}(\lambda)P(\lambda) = Q(\lambda)N^{-1}(\lambda)$$

such that $[J, B, C]$ is a minimal realization and M and P (resp. N and Q) are left (resp. right) coprime, then the columns of C form a full system of Jordan chains of M and the columns of B^T make up a full system of reversed chains of N^T.

PROOF. From $M(\lambda) = \sum_{\mu=0}^{m} A_\mu \lambda^\mu$ and $(\lambda I - J)^{-1} = \sum_{\nu=0}^{\infty} J^\nu \lambda^{-(\nu+1)}$ the expansion of

$$M(\lambda)C(\lambda I - J)^{-1}B \tag{3.4}$$

into a formal power series can be obtained. Since the matrix (3.4) is equal to the polynomial matrix P, the coefficients of λ^{-s}, $s = 1, 2, \ldots$, vanish. Therefore

$$\sum_{\mu=0}^{m} A_\mu C J^{\mu+\sigma} B = 0, \qquad \sigma = 0, 1, 2, \ldots,$$

and (2.6) yields $\sum_{\mu=0}^{m} A_\mu C J^\mu = 0$. Hence C satisfies the conditions (2.4) and (2.5). Lemma 3 implies (2.3). Analogous reasoning is valid for B.

COROLLARY [7]. *If $M \in C^{n\times n}[\lambda]$ has a proper rational inverse and $[J, B, C]$ is a minimal realization of M^{-1} such that*

$$M^{-1}(\lambda) = C(\lambda I - J)^{-1}B$$

then the columns of C (of B) form a full system of (reversed) Jordan chains of M (of M^T).

PROOF. The degree of the least common denominator of all minors of M^{-1} is the degree of the determinant of M. The result is thus a consequence of Lemma 2 and the preceding theorem.

4. The factorization theorem. For a nonsingular polynomial matrix $M \in \mathbf{C}^{n\times n}[\lambda]$ the quotient module $Y = \mathbf{C}^n[\lambda]/M\mathbf{C}^n[\lambda]$ is a $\mathbf{C}[\lambda]$ module and also a vector space of dimension r over $\mathbf{C}$, if $r = \deg \det M$. The module Y is important in linear system theory. We follow Fuhrmann [4] in describing a concrete representation of Y and defining a restricted shift operator in Y. Each n-vector of rational functions $f \in \mathbf{C}^n(\lambda)$ decomposes uniquely into a sum $f = g + h$ where the elements of g are proper rational and $h \in \mathbf{C}^n[\lambda]$.

Define π: $\mathbf{C}^n(\lambda) \to \mathbf{C}^n(\lambda)$ to be the map which discards the polynomial part h of f, $\pi f = g$. For $f \in \mathbf{C}^n[\lambda]$ put

$$\pi_M f = M\pi(M^{-1}f).$$

Then π_M: $\mathbf{C}^n[\lambda] \to \mathbf{C}^n[\lambda]$ is a projection map and $\ker \pi_M = M\mathbf{C}^n[\lambda]$. Thus $K_M := \operatorname{Im} \pi_M$, the image of π_M, is a $\mathbf{C}[\lambda]$ module isomorphic to Y. The shift operator $S(M)$ restricted to K_M is defined by

$$S(M)f(\lambda) = \pi_M \lambda f(\lambda), \qquad f \in K_M.$$

LEMMA 4 [4]. *The eigenvalues of $S(M)$ are the characteristic roots of M. The eigenvectors of $S(M)$ corresponding to an eigenvalue λ_0 have the form $(\lambda - \lambda_0)^{-1}M(\lambda)\xi$ and $M(\lambda_0)\xi = 0$.*

We establish a basis of K_M which exhibits $S(M)$ in Jordan form.

THEOREM 2. *If the columns of C are a full system of Jordan chains of M, then the columns of the polynomial matrix*

$$\hat{C}(\lambda) = M(\lambda)C(\lambda I - J)^{-1} \tag{4.1}$$

form a basis of K_M such that

$$S(M)\hat{C}(\lambda) = \hat{C}(\lambda)J.$$

PROOF. If we extend the definition of π in a natural way from $\mathbf{C}^n(\lambda)$ to rational matrices in $\mathbf{C}^{n\times k}(\lambda)$, then

$$\pi_M \hat{C}(\lambda) = M(\lambda)\pi C(\lambda I - J)^{-1} = \hat{C}(\lambda).$$

Therefore the columns of $\hat{C}(\lambda)$ are in K_M, and because of (2.5) and M nonsingular they are linearly independent. From $\pi\lambda(\lambda I - J)^{-1} = (\lambda I - J)^{-1}J$ we obtain

$$S(M)\hat{C}(\lambda) = M(\lambda)C\pi\lambda(\lambda I - J)^{-1} = \hat{C}(\lambda)J.$$

Analogously $\hat{B}(\lambda)$, defined by

$$\hat{B}(\lambda) = (\lambda I - J)^{-1}BM(\lambda), \tag{4.2}$$

is a polynomial matrix and the columns of $\hat{B}(\lambda)^T$ form a basis of K_{M^T} and $S(M^T)\hat{B}(\lambda)^T = \hat{B}(\lambda)^T J^T$.

The preceding definitions of $\hat{C}(\lambda)$ and $\hat{B}(\lambda)$ lead to the factorization result.

THEOREM 3. *Let $M \in \mathbf{C}^{n\times n}[\lambda]$ be a matrix which has a proper rational inverse and let M^{-1} have a minimal realization $[J, B, C]$,*

$$M^{-1}(\lambda) = C(\lambda I - J)^{-1}B,$$

such that J is an $r \times r$ matrix in Jordan form. Then

$$M(\lambda) = \hat{C}(\lambda)(\lambda I - J)\hat{B}(\lambda)$$

where $\hat{B}(\lambda)$ and $\hat{C}(\lambda)$ are defined by (4.1) *and* (4.2).

REMARK. In the special case of $M(\lambda) = \lambda I - A$ the theorem yields the transformation of A into Jordan normal form, that is $\lambda I - A = C(\lambda I - J)B$, $B = C^{-1}$.

In the sequel of this note we shall study the case where M does not have a proper rational inverse and the more general one of M being rectangular. It will be shown that a factorization of the form

$$M(\lambda) = \hat{C}(\lambda)(\lambda A_1 + A_2)\hat{B}(\lambda) \tag{4.3}$$

is possible and that Kronecker's theory of matrix pencils [5] can be extended to rectangular polynomial matrices. In the special case of a unimodular matrix M, (4.3) leads to $M(\lambda) = \hat{C}(\lambda)(\lambda R + I)\hat{B}(\lambda)$ where R is nilpotent.

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