A JORDAN FACTORIZATION THEOREM FOR POLYNOMIAL MATRICES

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Abstract. It is shown that a complex polynomial matrix \( M(\lambda) \) which has a proper rational inverse can be factored into \( M(\lambda) = \hat{C}(\lambda)(\lambda I - J)\hat{B}(\lambda) \)
where \( J \) is a matrix in Jordan normal form and the columns of \( \hat{C}(\lambda) \) consist of eigenvectors and generalized eigenvectors of a linear operator associated with \( M(\lambda) \). For a proper rational matrix \( W \) with factorizations \( W(\lambda) = C(\lambda I - J)^{-1}B = M(\lambda)^{-1}P(\lambda) = Q(\lambda)N(\lambda)^{-1} \) it will be proved that \( C \) consists of Jordan chains of \( M \) and \( B \) of Jordan chains of \( N \).

1. Introduction. Any linear polynomial matrix \( L(\lambda) = \lambda I - A, A \in \mathbb{C}^{n \times n} \), admits two distinct canonical factorizations. One is given by

\[
L(\lambda) = F(\lambda)D(\lambda)G(\lambda)
\]

where \( F \) and \( G \) are unimodular and \( D = \text{diag}(d_1, \ldots, d_n) \) is the Smith form of \( L \). The other is

\[
L(\lambda) = C(\lambda I - J)B
\]

where \( J \) is the Jordan form of \( A, B = C^{-1} \), and the columns of \( C \) are chains of eigenvectors and generalized eigenvectors of \( A \). In the case of a general complex polynomial matrix \( M \in \mathbb{C}^{n \times n}[\lambda], \)

\[
M(\lambda) = \sum_{\mu=0}^{m} A_\mu \lambda^\mu, \quad A_\mu \in \mathbb{C}^{n \times n},
\]

only a factorization of type (1.1) has been considered. It is the purpose of this note to show that \( M \) too can be factored into a product of type (1.2),

\[
M(\lambda) = \hat{C}(\lambda)(\lambda I - J)\hat{B}(\lambda),
\]

provided \( M \) has a proper rational inverse. The columns of \( \hat{C}(\lambda) \) consist of eigenvectors and generalized eigenvectors of a linear operator associated with \( M \). Our investigation uses Fuhrmann’s approach [4] to the realization of rational matrices.

2. Jordan chains of polynomial matrices. We shall use the following notation. \( \mathbb{C}[\lambda] \) will denote the space of all column vectors with complex polynomial entries. Similarly \( \mathbb{C}^{n \times k}[\lambda] \) denotes the vector space of all \( n \times k \) complex polynomial matrices. The entries of vectors in \( \mathbb{C}[\lambda] \) and of matrices in \( \mathbb{C}^{n \times k}[\lambda] \) are complex rational functions. A complex function \( f \) is called
proper rational, if it is the quotient of two polynomials, \( f = p/q \), and \( \deg p < \deg q \). An element of \( \mathbb{C}^n(\lambda) \) or \( \mathbb{C}^{n \times n}(\lambda) \) is proper rational if all of its entries are proper rational.

For the definitions and results of this section we refer to [1], [6], [7]. A number \( \lambda_0 \in \mathbb{C} \) is called a characteristic root of \( M \in \mathbb{C}^{n \times n}[\lambda] \), if \( \det M(\lambda_0) = 0 \). A Jordan chain of \( M \) corresponding to the characteristic root \( \lambda_0 \) is a sequence of vectors \( c_1, c_2, \ldots, c_s, c_s \in \mathbb{C}^n \), which satisfy \( c_1 \neq 0 \) and

\[
\sum_{\mu=0}^{j} \frac{1}{\mu!} M(\lambda_0)c_{j+1-p} = 0, \quad j = 0, 1, \ldots, s-1.
\]

The chain \( c_1, \ldots, c_s \) of length \( s \) will be called full, if it cannot be extended to a chain of length \( s + 1 \). The sequence \( c_s, c_{s-1}, \ldots, c_1 \) is called a reversed chain. In the special case of \( M(\lambda) = \lambda I - A, A \in \mathbb{C}^{n \times n} \), the relations (2.1) yield

\[
(\lambda_0 I - A)c_1 = 0, \quad (\lambda_0 I - A)c_{i+1} = c_i, \quad i = 1, \ldots, s-1,
\]

and a Jordan chain of \( M \) consists of an eigenvector \( c_1 \) and generalized eigenvectors of \( A \).

We shall use \( J \) for a matrix in (upper triangular) Jordan form,

\[
J = \text{block diag}(J_1, \ldots, J_g),
\]

where

\[
J_\gamma = \begin{bmatrix} 
\lambda_\gamma & 1 & 0 & \cdots & 0 \\
0 & \lambda_\gamma & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \lambda_\gamma 
\end{bmatrix}
\]

The columns of a matrix \( C \in \mathbb{C}^{n \times r} \) are said to form a full system [7] of Jordan chains of \( M(\lambda) = \sum_{\mu=0}^{m} A_\mu \lambda^\mu \), if

\[
r = \deg \det M
\]

and if there is a matrix \( J \in \mathbb{C}^{r \times r} \) such that

\[
\sum_{\mu=0}^{m} A_\mu CJ^\mu = 0
\]

and

\[
\text{rank } \begin{bmatrix} 
C \\
CJ \\
\vdots \\
CJ^{r-1} 
\end{bmatrix} = r.
\]

The conditions (2.3)–(2.5) are equivalent (see [7]) to the property that the columns of \( Ce^{Jt} \) form a fundamental system of solutions of the differential
equation
\[ \sum_{\mu=0}^{m} A_{\mu} x^{(\mu)}(t) = 0. \]

If \( J \) has the block structure (2.2), then \( C \) can be partitioned into \( C = (C_1, \ldots, C_q) \) such that the columns of each \( C_i \) form a full chain. Analogously the columns of \( B^T \) are said to form a \textit{full system of reversed chains} of \( M^T \) if \( \sum J^\mu B A_{\mu} = 0 \) and
\[ \text{rank}(B, JB, \ldots, J^{r-1}B) = r. \] (2.6)

In the case \( M(\lambda) = \lambda J - A \) the preceding definitions yield nonsingular matrices \( C \) and \( B \) such that
\[ \lambda J - A = C(\lambda I - J)C^{-1} = B^{-1}(\lambda J - J)B. \]

3. Realizations. Let \( W \in \mathbb{C}^{k \times n}(\lambda) \) be a proper rational matrix. Thus \( W \) has a formal expansion
\[ W(\lambda) = \sum_{\mu=0}^{\infty} W^\mu \lambda^{-\mu}. \]

**Lemma 1** [4]. There is an invertible matrix \( M \in \mathbb{C}^{k \times k}[\lambda] \) and a matrix \( P \in \mathbb{C}^{k \times n}[\lambda] \) such that
\[ W = M^{-1}P \] (3.1)
and \( M \) and \( P \) are left coprime. \( M \) and \( P \) are unique up to a common left unimodular factor. Analogously there exist an invertible matrix \( N \in \mathbb{C}^{n \times n}[\lambda] \) and a matrix \( Q \in \mathbb{C}^{k \times n}[\lambda] \) such that
\[ W = QN^{-1} \] (3.2)
and \( N \) and \( Q \) are right coprime.

**Definition** (see e.g. [2], [4]). A triple of linear operators \([A, B, C]\), \( B: \mathbb{C}^n \rightarrow X \), \( A: X \rightarrow X \), \( C: X \rightarrow \mathbb{C}^k \) is called a \textit{realization} of \( W \), if
\[ W_i = CA^iB, \quad i = 0, 1, 2, \ldots. \]
The realization is called \textit{minimal}, if the dimension of the state space \( X \) is minimal.

We note the following result on realizations of \( W \).

**Lemma 2** (see [3]). If \( p \) is the least common denominator of all minors of \( W \) and \( r = \deg p \), then there is a minimal realization \([J, B, C]\) of \( W \) with state space \( X = \mathbb{C}^r \) such that
\[ W(\lambda) = C(\lambda J - J)^{-1}B \] (3.3)
and \( J \in \mathbb{C}^{r \times r} \) is in Jordan form. For any minimal realization \( J, B, C \) the conditions (2.5) and (2.6) hold and the characteristic polynomial of \( J \) is equal to \( p \).

Since (2.5) is equivalent to \( \lambda J - J \) and \( C \) being right coprime and (2.6) to
\( \lambda I - J \) and \( B \) being left coprime, the last statement of Lemma 2 is contained in the following result.

**Lemma 3** [3, p. 102]. If \( P, Q \) and \( S \) are polynomial matrices such that \( S \) is left coprime to \( P \) and right coprime to \( Q \), then the least common denominator of all minors of \( W = QS^{-1}P \) is equal to \( \det S \).

The matrices \( C \) and \( B \) in (3.3) can be characterized in terms of Jordan chains of \( M \) and \( N \) in (3.1) and (3.2).

**Theorem 1.** If \( W \) has the representations

\[
W(\lambda) = C(\lambda I - J)^{-1}B = M^{-1}(\lambda)P(\lambda) = Q(\lambda)N^{-1}(\lambda)
\]

such that \([J, B, C]\) is a minimal realization and \( M \) and \( P \) (resp. \( N \) and \( Q \)) are left (resp. right) coprime, then the columns of \( C \) form a full system of Jordan chains of \( M \) and the columns of \( B^T \) make up a full system of reversed chains of \( N^T \).

**Proof.** From \( M(\lambda) = \sum_{\mu=0}^{\infty} A_\mu \lambda^{\mu} \) and \( (\lambda I - J)^{-1} = \sum_{s=0}^{\infty} J^s \lambda^{-(s+1)} \) the expansion of

\[
M(\lambda)C(\lambda I - J)^{-1}B
\]

into a formal power series can be obtained. Since the matrix (3.4) is equal to the polynomial matrix \( P \), the coefficients of \( \lambda^{-s}, s = 1, 2, \ldots \), vanish. Therefore

\[
\sum_{\mu=0}^{m} A_\mu CJ^{\mu+s}B = 0, \quad s = 0, 1, 2, \ldots,
\]

and (2.6) yields \( \sum_{\mu=0}^{m} A_\mu CJ^{\mu} = 0 \). Hence \( C \) satisfies the conditions (2.4) and (2.5). Lemma 3 implies (2.3). Analogous reasoning is valid for \( B \).

**Corollary [7].** If \( M \in C^{n \times n}[\lambda] \) has a proper rational inverse and \([J, B, C]\) is a minimal realization of \( M^{-1} \) such that

\[
M^{-1}(\lambda) = C(\lambda I - J)^{-1}B
\]

then the columns of \( C \) (of \( B \)) form a full system of (reversed) Jordan chains of \( M \) (of \( M^T \)).

**Proof.** The degree of the least common denominator of all minors of \( M^{-1} \) is the degree of the determinant of \( M \). The result is thus a consequence of Lemma 2 and the preceding theorem.

4. **The factorization theorem.** For a nonsingular polynomial matrix \( M \in C^{n \times n}[\lambda] \) the quotient module \( Y = C[\lambda]/MC[\lambda] \) is a \( \mathbb{Q}[\lambda] \) module and also a vector space of dimension \( r \) over \( \mathbb{C} \), if \( r = \deg \det M \). The module \( Y \) is important in linear system theory. We follow Fuhrmann [4] in describing a concrete representation of \( Y \) and defining a restricted shift operator in \( Y \). Each \( n \)-vector of rational functions \( f \in C^n(\lambda) \) decomposes uniquely into a sum \( f = g + h \) where the elements of \( g \) are proper rational and \( h \in C^n[\lambda] \).
Define $\pi: \mathbb{C}^n(\lambda) \to \mathbb{C}^n(\lambda)$ to be the map which discards the polynomial part $h$ of $f$, $\pi f = g$. For $f \in \mathbb{C}^n[\lambda]$ put

$$\pi_M f = M\pi(M^{-1}f).$$

Then $\pi_M: \mathbb{C}^n[\lambda] \to \mathbb{C}^n[\lambda]$ is a projection map and ker $\pi_M = M\mathbb{C}^n[\lambda]$. Thus $K_M := \text{Im } \pi_M$, the image of $\pi_M$, is a $\mathbb{C}[\lambda]$ module isomorphic to $Y$. The shift operator $S(M)$ restricted to $K_M$ is defined by

$$S(M)f(\lambda) = \pi_M f(\lambda), \quad f \in K_M.$$

**Lemma 4 [4].** The eigenvalues of $S(M)$ are the characteristic roots of $M$. The eigenvectors of $S(M)$ corresponding to an eigenvalue $\lambda_0$ have the form $(\lambda - \lambda_0)^{-1}M(\lambda)\xi$ and $M(\lambda_0)\xi = 0$.

We establish a basis of $K_M$ which exhibits $S(M)$ in Jordan form.

**Theorem 2.** If the columns of $C$ are a full system of Jordan chains of $M$, then the columns of the polynomial matrix

$$\hat{C}(\lambda) = M(\lambda)C(\lambda I - J)^{-1}$$

form a basis of $K_M$ such that

$$S(M)\hat{C}(\lambda) = \hat{C}(\lambda)J.$$

**Proof.** If we extend the definition of $\pi$ in a natural way from $\mathbb{C}^n(\lambda)$ to rational matrices in $\mathbb{C}^{n \times k}(\lambda)$, then

$$\pi_M \hat{C}(\lambda) = M(\lambda)\pi C(\lambda I - J)^{-1} = \hat{C}(\lambda).$$

Therefore the columns of $\hat{C}(\lambda)$ are in $K_M$, and because of (2.5) and $M$ nonsingular they are linearly independent. From $\pi \lambda(\lambda I - J)^{-1} = (\lambda I - J)^{-1}J$ we obtain

$$S(M)\hat{C}(\lambda) = M(\lambda)C\pi \lambda(\lambda I - J)^{-1} = \hat{C}(\lambda)J.$$

Analogously $\hat{B}(\lambda)$, defined by

$$\hat{B}(\lambda) = (\lambda I - J)^{-1}BM(\lambda), \quad (4.2)$$

is a polynomial matrix and the columns of $\hat{B}(\lambda)^T$ form a basis of $K_M^T$ and $S(M^T)\hat{B}(\lambda)^T = \hat{B}(\lambda)^TO^T$.

The preceding definitions of $\hat{C}(\lambda)$ and $\hat{B}(\lambda)$ lead to the factorization result.

**Theorem 3.** Let $M \in \mathbb{C}^{n \times n}[\lambda]$ be a matrix which has a proper rational inverse and let $M^{-1}$ have a minimal realization $[J, B, C]$,

$$M^{-1}(\lambda) = C(\lambda I - J)^{-1}B,$$

such that $J$ is an $r \times r$ matrix in Jordan form. Then

$$M(\lambda) = \hat{C}(\lambda)(\lambda I - J)\hat{B}(\lambda)$$

where $\hat{B}(\lambda)$ and $\hat{C}(\lambda)$ are defined by (4.1) and (4.2).

**Remark.** In the special case of $M(\lambda) = \lambda I - A$ the theorem yields the transformation of $A$ into Jordan normal form, that is $\lambda I - A = C(\lambda I - J)B$, $B = C^{-1}$. 

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In the sequel of this note we shall study the case where \( M \) does not have a proper rational inverse and the more general one of \( M \) being rectangular. It will be shown that a factorization of the form
\[
M(\lambda) = \hat{C}(\lambda)(\lambda A_1 + A_2)\hat{B}(\lambda)
\]  
(4.3)
is possible and that Kronecker's theory of matrix pencils [5] can be extended to rectangular polynomial matrices. In the special case of a unimodular matrix \( M \), (4.3) leads to \( M(\lambda) = \hat{C}(\lambda)(\lambda R + I)\hat{B}(\lambda) \) where \( R \) is nilpotent.

REFERENCES


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