PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 2, July 1979

APPROXIMATION OF THE BERGMAN NORM BY THE NORMS OF THE DIRECT PRODUCT OF TWO SZEGŐ SPACES

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ABSTRACT. Let G be a bounded regular region in the plane. Let $H_2^{1/2}(G)$ denote the Szegő space of G composed of analytic functions on G with finite norm

$$\left\{ \frac{1}{2\pi} \int_{\partial G} |f(z)|^2 \, |dz| \right\}^{1/2} < \infty.$$

We set $f(z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z)$ ($\varphi_j, \psi_j \in H_2^{1/2}(G)$). Then, we determine a necessary and sufficient condition for $f(z)$ to make the equality

$$\frac{1}{\pi} \iint_G |f(z)|^2 \, dx \, dy$$

$$= \min\left\{ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{\partial G} \varphi_j(z_1) \overline{\varphi_k(z_1)} \, |dz_1| \, \frac{1}{2\pi} \int_{\partial G} \psi_j(z_2) \overline{\psi_k(z_2)} \, |dz_2| \right\}$$

hold. The minimum is taken here over all analytic functions $\sum_{j=1}^{\infty} \varphi_j(z_1)\psi_j(z_2)$ on $G \times G$ satisfying $f(z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z)$ on G.

1. Introduction and statement of result. Let G denote an N-ply connected bounded regular region in the plane with boundary contours $\{C_\nu\}_{\nu=1}^{N}$. Let $B(G)$ and $H_2^{1/2}(G)$ denote the Bergman space and the Szegő space of G composed of analytic functions on G with finite norms

$$\left\{ \iint_G |f(z)|^2 \, dx \, dy \right\}^{1/2} < \infty \quad (z = x + iy)$$

and

$$\left\{ \frac{1}{2\pi} \int_{\partial G} |f(z)|^2 |dz| \right\}^{1/2} < \infty,$$

respectively. In the latter case, $f(z)$ means the Fatou boundary value of f at $z \in \partial G$. Let $\{Z_\nu(z)\, dz\}_{\nu=1}^{N-1}$ be a basis of analytic differentials on $\overline{G}$ which are real along ∂G such that

$$Z_\nu(z) = \int_{C_\nu} L(\zeta, z) \, d\zeta,$$

Received by the editors June 15, 1978.

AMS (MOS) subject classifications (1970). Primary 30A31; Secondary 30A04, 30A40.

Key words and phrases. Bergman and Szegő kernels, the direct product of two Szegő spaces, Dirichlet norm, approximation of the Bergman norm, generalized isoperimetric inequalities.

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$L(\zeta, z)$ is the adjoint L-kernel of the usual Bergman kernel $K(\zeta, \bar{z})$ on G. Cf. [6, Chapter 4]. Let $K^E(z, \bar{u})$ and $\hat{K}(z, \bar{u})$ denote the exact Bergman kernel and the Szegő kernel of G, respectively. Then, we have the identity

$$4\pi\hat{K}(z, \bar{u})^2 = K^E(z, \bar{u}) + \sum_{\nu=1}^{N-1} \sum_{\mu=1}^{N-1} C_{\nu,\mu} \overline{Z_\nu(u)}\, Z_\mu(z) \quad \text{on } G$$

for some uniquely determined constants $\{C_{\nu,\mu}\}$. D. A. Hejhal [2] established the positive definiteness of the real symmetric matrix $\|C_{\nu,\mu}\|$ by means of the Riemann theta function. For an elementary proof and a general result for that positive definiteness, see [3].

In [4], we obtained the following theorem:

THEOREM A [4, THEOREM 2.1]. *Any $f(z) \in B(G)$ can be represented by a series*

$$f(z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z) \quad \left(\varphi_j, \psi_j \in H_2^{1/2}(G)\right) \tag{1.1}$$

and the inequality

$$\frac{1}{\pi} \iint_G |f(z)|^2\, dx\, dy$$

$$< \min\left\{ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{\partial G} \varphi_j(z_1) \overline{\varphi_k(z_1)}\, |dz_1|\, \frac{1}{2\pi} \int_{\partial G} \psi_j(z_2) \overline{\psi_k(z_2)}\, |dz_2| \right\} \tag{1.2}$$

is valid. The minimum is taken here over all analytic functions $\sum_{j=1}^{\infty}\varphi_j(z_1)\psi_j(z_2)$ on $G \times G$ satisfying (1.1).

Conversely, if the jk sum in (1.2) *is finite, then the function $f(z)$ defined by the series* (1.1) *belongs to the class $B(G)$.*

In order to state a sense of the jk sum in (1.2), we introduce the direct product

$$H = H_2^{1/2}(G) \otimes H_2^{1/2}(G)$$

of the two Szegő spaces. Let $\{\Phi_j(z)\}_{j=1}^{\infty}$ be a complete orthonormal system of $H_2^{1/2}(G)$. Then, the H is formed by functions on $G \times G$ such that

$$f(z_1, z_2) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \Phi_j(z_1)\Phi_k(z_2), \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |A_{j,k}|^2 < \infty \tag{1.3}$$

and the scalar product $(\ ,\ )_H$ of H is defined as follows:

$$\left(f(z_1, z_2), h(z_1, z_2)\right)_H = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \overline{B_{j,k}}, \tag{1.4}$$

where $h(z_1, z_2) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} B_{j,k}\Phi_j(z_1)\Phi_k(z_2) \in H$. The jk sum in (1.2) is the square of the norm of $f(z_1, z_2)$ in H. Cf. [1].

Next, we introduce the Hilbert space H_R which is formed by restricting the functions in H to the diagonal set D of $G \times G$ formed by all the elements

$\{(z, z); z \in G\}$. Here, for any such restriction $f \in H_R$, the norm $\|f\|_R$ is defined by $\min\|h\|_H$ for all $h \in H$, the restriction of which to D is f. Of course, $\| \ \|_H$ denotes the norm of H. Cf. [1, Theorem II, p. 361]. Hence, the minimum in (1.2) is the square of the norm of f in H_R. As a result of the general theory, we obtained the following:

THEOREM B [4, THEOREM 3.1, THE FIRST PART]. *In* (1.2), *equality holds for* $f(z) \in B(G)$ *if and only if*

$$(f(z), g(z))_R = 0 \quad \text{for all } g(z) \in B_{\text{real}}(G). \tag{1.5}$$

Here $(\ , \ )_R$ *denotes the scalar product of* H_R *and* $B_{\text{real}}(G)$ *the vector space generated by* $\{Z_\nu(z)\}_{\nu=1}^{N-1}$.

In this paper, we show that

THEOREM 1.1. *In* (1.2), *equality holds for* $f(z) \in B(G)$ *if and only if* $f(z)\,dz$ *is exact.*

Therefore this paper also gives supplementary remarks to [4]. In the proof of this theorem, the following theorem is important:

THEOREM C [5, THEOREM 2.1]. *For any*

$$f(z_1, z_2) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \Phi_j(z_1) \Phi_k(z_2) \in H,$$

the restriction $f(z, z)$ *to* D *can be uniquely decomposed as follows*:

$$f(z, z) = h'(z) + \sum_{\nu=1}^{N-1} d_\nu Z_\nu(z) \quad \text{on } G, \tag{1.6}$$

$h'(z) \in B(G)$ *and* $\{d_\nu\}$ *are constants and the inequality*

$$\begin{aligned}
&\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |A_{j,k}|^2 \\
&\quad > \min\left\{ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{\partial G} \varphi_j(z_1) \overline{\varphi_k(z_1)}\, |dz_1|\, \frac{1}{2\pi} \int_{\partial G} \psi_j(z_2) \overline{\psi_k(z_2)}\, |dz_2| \right\} \\
&\quad = \frac{1}{\pi} \iint_G |h'(z)|^2\, dx\, dy + \sum_{\nu=1}^{N-1} \sum_{\mu=1}^{N-1} D_{\nu,\mu} d_\nu \overline{d_\mu},
\end{aligned} \tag{1.7}$$

$$\|D_{\nu,\mu}\| \text{ is the inverse of } \|C_{\nu,\mu}\|$$

is valid. The minimum is taken here over all analytic functions $\sum_{j=1}^{\infty} \varphi_j(z_1)\psi_j(z_2)$ *on* $G \times G$ *satisfying*

$$f(z, z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z) \text{ on } G, \quad \varphi_j, \psi_j \in H_2^{1/2}(G). \tag{1.8}$$

2. Proof of Theorem. For any $f(z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z) = h'(z) + \sum_{\nu=1}^{N-1} d_\nu Z_\nu(z) \in B(G)$ $(\varphi_j, \psi_j \in H_2^{1/2}(bG))$, from Theorem A and Theorem B, we have

$$\begin{aligned}
&\frac{1}{\pi}\iint_G |f(z)|^2\,dx\,dy\\
&=\frac{1}{\pi}\iint_G \left|h'(z)+\sum_{\nu=1}^{N-1} d_\nu Z_\nu(z)\right|^2 dx\,dy\\
&< \min\left\{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2\pi}\int_{\partial G}\varphi_j(z_1)\overline{\varphi_k(z_1)}\,|dz_1|\frac{1}{2\pi}\int_{\partial G}\psi_j(z_2)\overline{\psi_k(z_2)}\,|dz_2|\right\}\\
&=\frac{1}{\pi}\iint_G |h'(z)|^2\,dx\,dy+\sum_{\nu=1}^{N-1}\sum_{\mu=1}^{N-1} D_{\nu,\mu}d_\nu\overline{d_\mu}. \qquad (2.1)
\end{aligned}$$

Since

$$\iint_G h'(z)\overline{Z_\nu(z)}\,dx\,dy=0 \quad \text{for } \nu=1,2,\ldots,N-1, \qquad (2.2)$$

(cf. [6, Chapter 4]), we thus have

$$\sum_{\nu=1}^{N-1}\sum_{\mu=1}^{N-1}\left(D_{\nu,\mu}-\iint_G Z_\nu(z)\overline{Z_\mu(z)}\,dx\,dy\right)d_\nu\overline{d_\mu}>0. \qquad (2.3)$$

Suppose that for $f(z)\in B(G)$, equality holds in (1.2) and so in (2.1), then from Theorem B and Theorem C, we have

$$\begin{aligned}
(f(z),g(z))_R &= \left(h'(z)+\sum_{\nu=1}^{N-1} d_\nu Z_\nu(z), g(z)\right)_R\\
&=\left(\sum_{\nu=1}^{N-1} d_\nu Z_\nu(z), g(z)\right)_R\\
&=0 \quad \text{for all } g(z)\in B_{\text{real}}(G). \qquad (2.4)
\end{aligned}$$

Hence all the d_ν are zero and so we have the desired result.

In particular, in the above proof, we obtained the following theorem. Cf. [2, Theorem 39, p. 107]:

THEOREM 2.1. *The matrix*

$$\left\| D_{\nu,\mu}-\iint_G Z_\nu(z)\overline{Z_\mu(z)}\,dx\,dy\right\|^{(N-1)(N-1)}$$

is positive definite.

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