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PROOF OF A CONJECTURE OF DOOB

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ABSTRACT. Let $\mathfrak{D}(\rho)$ be the class of all holomorphic functions f in the unit disc Δ such that $f(0)=0$ and there exists an arc $\Upsilon_f \subsetneq \partial\Delta$ with length $|\Upsilon_f| > \rho$ such that $\underline{\lim}|f(z)| > 1$, $z \to \tau \in \Upsilon_f$. In 1935, J. L. Doob asked, in essence, whether the Bloch norms $\{\|f\| = \sup_{z\in\Delta}|f'(z)|(1-|z|^2)\}$ have a positive lower bound for the class $\mathfrak{D}(\rho)$. We show that if $f \in \mathfrak{D}(\rho)$ there exists a $z_f \in \Delta$ such that

$$|f'(z_f)|\left(1-|z_f|^2\right) > \frac{2}{e}\ \frac{\sin(\pi-\rho/2)}{(\pi-\rho/2)}.$$

1. Introduction. For fifty years the problem of determining precise estimates for the Bloch and Landau constants has been pursued actively. We describe the problems with the unit disc Δ in the complex plane as the domain for all functions which are assumed to be holomorphic unless otherwise specified. The Bloch number of a function f is

$$b(f) = \sup\{r \mid \text{there exists a subdomain } \Delta_1 \subseteq \Delta \text{ such that } f \text{ is univalent on } \Delta_1 \text{ and } f(\Delta_1) \text{ contains a (Euclidean) disc of radius } r\},$$

and the Landau number of f is

$$l(f) = \sup\{r \mid f(\Delta) \text{ contains a (Euclidean) disc of radius } r\}.$$

For a family of functions $\mathfrak{F}$ the corresponding constants are

$$b(\mathfrak{F}) = \inf b(f),\ \ f \in \mathfrak{F}; \qquad l(\mathfrak{F}) = \inf l(f),\ \ f \in \mathfrak{F}.$$

Of course $b(\mathfrak{F}) \leqslant l(\mathfrak{F})$. Classically the family $\mathfrak{F}$ was taken to be either $\mathfrak{B}$, the family of all f with $|f'(0)| \geqslant 1$, or $\mathfrak{S}$, the family of all univalent f with $|f'(0)| \geqslant 1$. The best estimates are

$$\sqrt{3}/4 = .433\ldots < b(\mathfrak{B}) < .472 \leqslant \cdots \leqslant .50 < l(\mathfrak{B}) < .544 < \cdots < b(\mathfrak{S}) < .658.$$

The lower estimate for $b(\mathfrak{B})$ is by Heins [12], improving Ahlfors' estimate [1] $b(\mathfrak{B}) \geqslant \sqrt{3}/4$; the upper estimate for $b(\mathfrak{B})$ is by Ahlfors and Grunsky [2];

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the corresponding estimates for $l(\mathfrak{B})$ are by Pommerenke [15] improving Ahlfors [1] and Robinson as reported by Hayman [11]. For $b(\mathfrak{S})$ the estimates are by Landau [13] and Robinson [16]. Other families of functions have been studied in this connection (see, for example, Hayman [10]).

In a series of papers begun in 1932, Doob [5]–[9] considered certain boundary value problems as well as Bloch and Landau constants for a class of functions with a different normalization from $\mathfrak{B}$. For any $0 < \rho < 2\pi$, Doob defined $\mathfrak{D}(\rho)$ as the family of all functions f with $f(0) = 0$ such that there exists an open arc Υ_f (depending on f) in $\partial\Delta$ with length $|\Upsilon_f| \geqslant \rho$ and such that $\underline{\lim}_{z\to\tau}|f(z)| \geqslant 1$ for each $\tau \in \Upsilon_f$. In [7] Doob showed that $b(\mathfrak{D}(\rho))$ is positive for each $0 < \rho < 2\pi$ and also that there exists a positive constant $a(\rho)$ such that $f \in \mathfrak{B}$ implies $a(\rho)f \in \mathfrak{D}(\rho)$. More precisely he showed that $a(\rho)f \circ g \in \mathfrak{D}(\rho)$ for some function g, $|g| < 1$, $g(0) = 0$, where g depends on f. The reverse implication was thought by Doob to be false. If L is a Möbius transformation of Δ onto Δ, then it is clear that $b(f \circ L) = b(f)$, $l(f \circ L) = l(f)$, and $(f \circ L)'(0) = (1 - |L(0)|^2)f'(L(0))$. Doob's question now becomes whether or not $f \in D(\rho)$ implies that there exists a positive constant $\sigma(\rho)$ independent of f and a point $z_f \in \Delta$ such that $|f'(z_f)|(1 - |z_f|^2) > \sigma(\rho)$. If such a constant does exist, it follows that $(\sigma(\rho))^{-1}(f \circ L) \in \mathfrak{B}$ for any L such that $L(0) = z_f$. Professor Doob brought this question to our attention several months ago.

2. Main result. Definition. For $\Upsilon \subseteq \partial\Delta$ let $\omega(z, \Upsilon, \Delta)$ denote the harmonic measure at z of Υ relative to Δ, and set

$$S(\rho, \Upsilon) = \{z \in \Delta | \omega(z, \Upsilon, \Delta) \geqslant \rho/2\pi\}, \qquad 0 < \rho < 2\pi,$$

and let

$$\sigma(\rho) = \frac{2}{e}\ \frac{\sin(\pi - \rho/2)}{(\pi - \rho/2)}.$$

Our proof is based upon a sharpened form of the Lehto-Virtanen differential two constant theorem [14] due to S. Dragosh and D. C. Rung [4]. Because this result is unpublished we give a simplified version of the result sufficient for the main result.

Theorem D–R. *Let f be a meromorphic function in D. For some fixed $S(\rho, \Upsilon)$ suppose that*

(i) $\sup_{z\in S(\rho,\Upsilon)}|f(z)| = M < \infty$;

(ii) *for each* $\tau \in \Upsilon$, $\overline{\lim}_{z\to\tau}|f(z)| \leqslant m < M$;

(iii) *there exists a* $q \in \partial S(\rho, \Upsilon) \cap \Delta$ *at which* $|f(q)| = M$.

Then

$$|f'(q)|\left(1 - |q|^2\right) \geqslant e\sigma(\rho)M \log(M/m).$$

PROOF. According to the two-constant theorem with $\omega(z) = \omega(z, \Upsilon, S(\rho, \Upsilon))$ we have for $z \in S(\rho, \Upsilon)$, $|f(z)| \leq M(m/M)^{\omega(z)}$, with equality occurring at $z = q$. The level line $|f(z)| = M$ is tangent to $\partial S(\rho, \Upsilon)$ at $z = q$ and so $|f'(q)| \neq 0$. If n is the inner normal to $\partial S(\rho, \Upsilon)$ at q then

$$\left.\frac{\partial|f(z)|}{\partial n}\right|_{z=q} \leq \left.\frac{\partial\omega(z)}{\partial n}\right|_{z=q} M \log \frac{m}{M}.$$

It is easy to calculate that

$$\left.\frac{\partial|f(z)|}{\partial n}\right|_{z=q} = -|f'(q)| \quad \text{and} \quad \left.\frac{\partial\omega(z)}{\partial n}\right|_{z=q} = \frac{e\sigma(\rho)}{1-|q|^2}.$$

If these are substituted in the above inequality the result is proved.

THEOREM 1. *If* $f \in (\rho_0)$, $0 < \rho_0 < 2\pi$, *there exists at least one point* $z_f \in S(\rho_0, \Upsilon_f)$ *at which* $|f'(z_f)|(1 - |z_f|^2) \geq \sigma(\rho_0)$.

PROOF. To the contrary suppose

$$|f'(z)|\left(1 - |z|^2\right) < \sigma(\rho_0) \tag{1.0}$$

for all $z \in S(\rho_0, \Upsilon_f)$ and consider the meromorphic function $g = 1/f$. For a subarc $\Upsilon^* \subseteq \Upsilon_f$ let $M(\Upsilon^*) = \sup|g(z)|$, $z \in S(\rho_0, \Upsilon^*)$. (We allow $M(\Upsilon^*) = \infty$ if g has a pole in $S(\rho_0, \Upsilon^*)$.) Theorem D–R given above implies that at a point $q \in \Delta \cap \partial S(\rho_0, \Upsilon^*)$ at which $|g(q)| = M(\Upsilon^*) < \infty$ we have

$$e\sigma(\rho_0)M(\Upsilon^*)\log M(\Upsilon^*) \leq |g'(q)|\left(1 - |q|^2\right). \tag{1.1}$$

After a brief calculation, assumption (1.0) gives from (1.1) that

$$M(\Upsilon^*)\log M(\Upsilon^*) < (1/e)\left(M(\Upsilon^*)\right)^2$$

or

$$\left(M(\Upsilon^*)\right)^{-1}\log M(\Upsilon^*) < 1/e. \tag{1.2}$$

Because the function $x^{-1}\log x$ has a maximum value of $1/e$ which occurs when $x = e$, the inequality (1.2) says that $M(\Upsilon^*)$ lies in one of the intervals $[1, e)$ or $(e, \infty]$. Also, since g is a continuous function, the possible values of $M(\Upsilon^*)$ form a connected set. Because there exist small arcs Υ^* for which $M(\Upsilon^*) < e$, we must have $M(\Upsilon^*) < e$ for each arc Υ^*. But $M(\Upsilon_f) = \infty$ since $0 \in \partial S(\rho_0, \Upsilon_f)$ and $g(0) = \infty$. This contradiction proves the theorem.

The constant $\sigma(\rho)$ is best possible, at least in the limiting case $\sigma(\rho) \to 2/e$ as $\rho \to 2\pi$. The functions $f_n(z) = z^n$ all are in $\mathfrak{D}(\rho)$ for all $0 < \rho < 2\pi$. A brief calculation shows that $|f_n'(z)|(1 - |z|^2)$ has a maximum value of $2(n/(n + 1))[1 - 2/(n + 1)]^{(n-1)/2}$ which tends to $2/e$ as $n \to \infty$.

This theorem yields a new approach to upper bounds for $b(\mathfrak{B})$, in that $f \in \mathfrak{D}(\rho)$ implies that $b(\mathfrak{B}) \leq b(f)(\sigma(\rho))^{-1}$ and a similar inequality for $l(\mathfrak{B})$. For example,

$$f_n(z) = \frac{e^{n((z+1)/(z-1))} - e^{-n}}{1 - e^{n((z+1)/(z-1))}e^{-n}}$$

is in $\mathfrak{D}(\rho)$ for each $n = 1, 2, \ldots,$ and any $0 < \rho < 2\pi$. Because f_n omits $-e^{-n}$, $l(f_n) = (1 + e^{-n})/2$ and so $l(B) \leqslant (\sigma(\rho))^{-1}((1 + e^{-n})/2)$. Letting $\rho \to 2\pi$ and $n \to \infty$ gives $l(\mathfrak{B}) \leqslant e/4 \simeq .679$. Whether better estimates can be obtained awaits further study.

References

1. L. V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), 359–364.
2. L. V. Ahlfors and H. Grunsky, *Über die Blochsche Konstante*, Math. Z. **42** (1937), 671–673.
3. A. Bloch, *Les théorèmes de M. Valiron sur les fonctions entières, et la théorie de l'uniformisation*, Ann. Fac. Sci. Univ. Toulouse **3** (1925), 1–22.
4. S. Dragosh and D. C. Rung, *Normal functions bounded on arcs and a proof of the Gross cluster-value theorem*, Hiroshima Math. J. (to appear).
5. J. L. Doob, *The boundary values of analytic functions*, Trans. Amer. Math. Soc. **34** (1932), 153–170.
6. ______, *The boundary values of analytic functions*. II, Trans. Amer. Math. Soc. **35** (1933), 718–751.
7. ______, *The ranges of analytic functions*, Ann. of Math. **36** (1935), 117–126.
8. ______, *Cluster values of sequences of analytic functions*, Sankhyā Ser. A **25** (1963), 137–148.
9. ______, *One sided cluster value theorems*, Proc. London Math. Soc. **3** (1963), 461–470.
10. W. K. Hayman, *Some applications of the transfinite diameter to the theory of functions*, J. Analyse Math. **1** (1951), 155–179.
11. ______, *Research problems in function theory*, Athlone Press, Univ. of London, London, 1967.
12. M. Heins, *A class of conformal metrics*, Bull. Amer. Math. Soc. **67** (1961), 475–478.
13. E. Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Math. Z. **30** (1929), 608–634.
14. O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.
15. C. Pommerenke, *Bloch functions*, J. London Math. Soc. **2** (1970), 689–695.
16. R. M. Robinson, *The Bloch constant $\mathfrak{A}$ for a schlicht function*, Duke Math. J. **1** (1935), 535–540.

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