

A PROBLEM OF GEOMETRY IN R^n

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ABSTRACT. Let \mathcal{F} be a finite family of at least $n + 1$ convex sets in the n -dimensional Euclidean space R^n . Helly's theorem asserts that if all the $(n + 1)$ -subfamilies of \mathcal{F} have nonempty intersection, then \mathcal{F} also has nonempty intersection. The main result in this paper is that if almost all of the $(n + 1)$ -subfamilies of \mathcal{F} have nonempty intersection, then \mathcal{F} has a subfamily with nonempty intersection containing almost all of the sets in \mathcal{F} .

1. Introduction. A family \mathcal{F} of sets is said to be an I -family if $\bigcap \mathcal{F} \neq \emptyset$. Let \mathcal{F}_i denote the collection of I -subfamilies of \mathcal{F} of size i . In this notation, the classic result of Helly [5] may be stated as follows:

HELLY'S THEOREM. *If \mathcal{F} is a family of x convex sets in R^n with $x > n$, then \mathcal{F} is an I -family if $|\mathcal{F}_{n+1}| = \binom{x}{n+1}$. Counterexamples show that the conclusion is false if $|\mathcal{F}_{n+1}| < \binom{x}{n+1}$.*

In this paper, we study the maximal I -subfamily of \mathcal{F} when it falls short of the entire family. Greek letters α , ρ and ω always denote real numbers and, unless otherwise stated, we assume that $0 < \alpha, \rho, \omega < 1$.

Let \mathcal{F} be a family of x compact convex sets in R^n . The compactness condition is introduced only for convenience as we deal with finite families of sets. We assume that $|\mathcal{F}_r| > \alpha \binom{x}{r}$ for some α and r , $n < r < x$. We ask the following questions:

- (A) What is the maximal size ρx of an I -subfamily of \mathcal{F} ?
- (B) Does ρ tend to 1 as α tends to 1?

In R^1 with $r = 2$, Abbott and Katchalski [1] proved that $\rho = 1 - \sqrt{1 - \alpha}$ and that this result is best possible. The answer to question (B) is thus in the affirmative.

In this paper, we shall deal with the problem in R^n in a more general setting. We consider families of p -tuplets in R^n , which are defined as unions of p compact convex sets in R^n . Our main tools are two lemmas which are given in the next section. In §3, we give a lower bound for ρ in answer to question (A).

The special case $p = 1$ is explored further in §4, leading to an affirmative answer to question (B) for compact convex sets in R^n . For p -tuplets with $p > 2$, the answer to question (B) is essentially negative. This is pointed out at the end of §3.

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For other related problems, consult the comprehensive paper of Danzer, Grünbaum and Klee [2] and the excellent bibliography section in Hadwiger, Debrunner and Klee [4]. For results on p -tuplets, see Grünbaum and Motzkin [3], Katchalski and Liu [6] and Larman [7].

2. Two lemmas. A family \mathcal{F} of x sets is said to have property (n) , $n < x$, if \mathcal{F}_n is nonempty and there exists a function f mapping \mathcal{F}_n into the subsets of $\cup \mathcal{F}$ such that:

- (1) $f(\mathcal{Q}) \cap (\cap \mathcal{Q}) \neq \emptyset$ for all $\mathcal{Q} \in \mathcal{F}_n$;
- (2) if \mathcal{B} is an I -subfamily of \mathcal{F} , $|\mathcal{B}| > n$, then there exists $\mathcal{Q} \in \mathcal{F}_n$ such that $\mathcal{Q} \subset \mathcal{B}$ and $f(\mathcal{Q}) \cap (\cap \mathcal{B}) \neq \emptyset$.

COMBINATORIAL LEMMA. *Let \mathcal{F} be a family of x sets with property (n) with $w = \max\{|f(\mathcal{Q})|: \mathcal{Q} \in \mathcal{F}_n\}$. If $|\mathcal{F}_r| > \alpha \binom{x}{r}$ for some α and r , $n < r < x$, then \mathcal{F} has an I -subfamily of size at least t , where t is the smallest integer for which $\binom{t-n}{r-n} \binom{x}{n} > \alpha \binom{x}{r} / w$.*

PROOF. Define $g: \mathcal{F}_r \rightarrow \mathcal{F}_n$ in such a way that for $\mathcal{B} \in \mathcal{F}_r$, $g(\mathcal{B}) \subset \mathcal{B}$ with $f \circ g(\mathcal{B}) \cap (\cap \mathcal{B}) \neq \emptyset$. Since \mathcal{F} has property (n) , $g(\mathcal{B})$ can always be chosen. If more than one choice is possible, a random selection is made.

Now $|\mathcal{F}_r| > \alpha \binom{x}{r}$ while $|\mathcal{F}_n| < \binom{x}{n}$. Hence for some $\mathcal{Q} \in \mathcal{F}_n$, $f(\mathcal{Q}) \cap (\cap \mathcal{B}) \neq \emptyset$ for at least $\alpha \binom{x}{r} / \binom{x}{n}$ of the \mathcal{B} 's in \mathcal{F}_r . Now $|f(\mathcal{Q})| < w$. Hence for some $z \in f(\mathcal{Q})$, z belongs to $\cap \mathcal{B}$ for at least $\alpha \binom{x}{r} / w \binom{x}{n}$ of these \mathcal{B} 's. Let z belong to k of the sets in \mathcal{F} . Then it can belong to $\cap \mathcal{B}$ for at most $\binom{k-n}{r-n}$ of the \mathcal{B} 's in \mathcal{F}_r . Hence we must have $k > t$ with t as given in the lemma. The lemma follows immediately. \square

Now let A be any nonempty compact subset of R^n . Define $h(A)$ to be the point $(a_1, a_2, \dots, a_n) \in A$ where

$$a_1 = \{\max x_1: (x_1, x_2, \dots, x_n) \in A\},$$

and for $2 \leq i \leq n$,

$$a_i = \{\max x_i: (a_1, \dots, a_{i-1}, x_i, \dots, x_n) \in A\}.$$

Let \odot be the lexicographical order on R^n , that is, $(a_1, a_2, \dots, a_n) \odot (b_1, b_2, \dots, b_n)$ if $a_i = b_i$ for $1 \leq i < n$ or if there exists some $k < n$ such that $a_i = b_i$ for $i < k$ and $a_k > b_k$. Note that $h(A) \odot h(B)$ if $A \supset B$, and that $h([a, b]) = b$ in R^1 .

LEXICOGRAPHICAL LEMMA. *Let \mathcal{B} be a finite I -family of compact convex sets in R^n with $|\mathcal{B}| > n$. Then \mathcal{B} has a subfamily \mathcal{Q} , $|\mathcal{Q}| = n$, such that $h(\cap \mathcal{Q}) = h(\cap \mathcal{B})$.*

PROOF. Let $h(\cap \mathcal{B}) = (a_1, a_2, \dots, a_n)$. Define a subset D of R^n by $D = \{(x_1, x_2, \dots, x_n): x_1 > a_1\} \cup (\cup_{i=2}^n \{(a_1, \dots, a_{i-1}, x_i, \dots, x_n): x_i > a_i\})$. It is easy to see that D is convex and that $D \cap (\cap \mathcal{B}) = \emptyset$.

Let $\mathcal{B}^* = \mathcal{B} \cup \{D\}$. If every subfamily of \mathcal{B}^* of size $n + 1$ is an I -subfamily, it will follow from Helly's theorem that \mathcal{B}^* is an I -family too,

which it is not. Hence some subfamily \mathcal{Q}^* of \mathcal{B}^* of size $n + 1$ has empty intersection. Clearly $D \in \mathcal{Q}^*$. Let $\mathcal{Q} = \mathcal{Q}^* - \{D\}$. Now $|\mathcal{Q}| = n$ and $D \cap (\cap \mathcal{Q}) = \emptyset$.

It follows from the definition of D that $h(\cap \mathcal{Q}) \otimes h(\cap \mathcal{B})$. On the other hand, $(\cap \mathcal{Q}) \supset (\cap \mathcal{B})$ and $h(\cap \mathcal{Q}) \otimes h(\cap \mathcal{B})$. This proves the lemma. \square

3. p -tuplets in R^n . We now state and prove our general result on p -tuplets.

THEOREM A. *For each α , there is a ρ and an x_0 such that if \mathcal{F} is a family of x p -tuplets in R^n with $x \geq x_0$ and $|\mathcal{F}_r| \geq \alpha \binom{x}{r}$ for some r , $n < r < x$, then \mathcal{F} has an I -subfamily of size ρx . Furthermore, $\rho \geq (\alpha/p^n \binom{x}{n})^{1/(r-n)}$.*

PROOF. We first show that \mathcal{F} has property (n) . \mathcal{F}_n is clearly nonempty. We now verify the two conditions:

(1) For $\mathcal{Q} \in \mathcal{F}_n$, define $f(\mathcal{Q}) = \{h(A) : A \text{ a component of } \cap \mathcal{Q}\}$. By the definition of h , $f(\mathcal{Q}) \cap (\cap \mathcal{Q}) \neq \emptyset$. In fact, $f(\mathcal{Q}) \subset (\cap \mathcal{Q})$. We point out that $|f(\mathcal{Q})| < p^n$ for all $\mathcal{Q} \in \mathcal{F}_n$.

(2) Let \mathcal{B} be any I -subfamily of \mathcal{F} , $|\mathcal{B}| \geq n$. Let B be the component of $\cap \mathcal{B}$ which contains $h(\cap \mathcal{B})$. For any $F \in \mathcal{B}$, let \bar{F} be the component of F which contains B and let $\bar{\mathcal{B}} = \{\bar{F} : F \in \mathcal{B}\}$. By the lexicographical lemma, there exists $\mathcal{Q} \subset \bar{\mathcal{B}}$ such that $|\mathcal{Q}| = n$ and $h(\cap \mathcal{Q}) = h(\cap \bar{\mathcal{B}}) = h(B) \in B$. Now $\mathcal{Q} = \{F : \bar{F} \in \mathcal{Q}\} \in \mathcal{F}_n$ is contained in \mathcal{B} and $f(\mathcal{Q}) \cap (\cap \mathcal{B}) \supset \{h(B)\} \neq \emptyset$.

By the combinatorial lemma, \mathcal{F} has an I -subfamily of size at least t , where t is the smallest integer for which $\binom{t-n}{r-n} \binom{x}{n} > \alpha \binom{x}{r}/p^n$. A crude estimation yields the desired result when $x \geq x_0$. \square

Theorem A is a generalization of an earlier result in R^1 of Katchalski and Liu [6]. In the same paper, it was proved that in R^1 with $p = 2$, $\rho < (r - 1)/r$ even if we allow $\alpha = 1$. Hence the answer to question (B) for p -tuplets is negative for $p > 2$, unless r is sufficiently large, for then the lower bound $(\alpha/p^n \binom{x}{n})^{1/(r-n)}$ is close to 1 if α is.

4. Compact convex sets in R^n . In this section, we restrict our attention to compact convex sets in R^n . We state the particular case $p = 1$ of Theorem A as:

THEOREM B. *For each α , there are a ρ and an x_0 such that if \mathcal{F} is a family of x compact convex sets in R^n with $x \geq x_0$ and $|\mathcal{F}_r| \geq \alpha \binom{x}{r}$ for some r , $n < r < x$, then \mathcal{F} has an I -subfamily of size ρx . Furthermore, $\rho \geq (\alpha/\binom{x}{n})^{1/(r-n)}$.*

As it stands, r being fixed, Theorem B does not imply that ρ tends to 1 as α does. We shall improve the lower bound via a third lemma.

STEPPING-UP LEMMA. *Let \mathcal{F} be a family of x convex sets in R^n such that $|\mathcal{F}_r| \geq \alpha \binom{x}{r}$ for some α and r , $n < r < x$. Then for any m , $r < m < x$, $|\mathcal{F}_m| \geq (1 - (1 - \alpha)\binom{x}{r})\binom{x}{m}$.*

PROOF. The number of subfamilies of \mathcal{F} of size r which do not belong to \mathcal{F}_r is at most $(1 - \alpha)\binom{x}{r}$. The number of subfamilies of \mathcal{F} of size m containing at least one of these subfamilies of size r is at most $(1 - \alpha)\binom{x}{r}\binom{x-r}{m-r} = (1 - \alpha)\binom{x}{m}\binom{m}{r}$. Since $m > r \geq n + 1$, Helly's theorem shows that the remaining subfamilies of size m are in \mathcal{F}_m , and there are at least $(1 - (1 - \alpha)\binom{m}{r})\binom{x}{m}$ of them. This proves the lemma. \square

We are now in a position to prove our main result.

THEOREM C. For each ρ , there is an α such that if \mathcal{F} is a family of x compact convex sets in R^n with $|\mathcal{F}_r| \geq \alpha\binom{x}{r}$ for some r , $n < r < x$, then \mathcal{F} has an I -subfamily of size ρx .

PROOF. Choose $m > r$ such that

$$\binom{m}{n}^{1/(n-m)} > 1 - \frac{1 - \rho}{2}$$

and also

$$\left(\frac{1 + \rho}{2}\right)^{1+1/(m-n)} > \rho.$$

Once chosen, m is fixed. Let $\bar{x} = \max\{m, x_0\}$ where x_0 is as in Theorem B.

Let $\alpha > \max\{1 - (1 - \rho)/2\binom{m}{r}, 1 - 1/\binom{\bar{x}}{r}\}$. We consider two cases:

(i) $x < \bar{x}$. We have

$$|\mathcal{F}_r| \geq \alpha\binom{x}{r} > \left(1 - 1/\binom{\bar{x}}{r}\right)\binom{x}{r} = \binom{x}{r} - \binom{x}{r}/\binom{\bar{x}}{r}.$$

It follows that $|\mathcal{F}_r| = \binom{x}{r}$ and Helly's theorem shows that \mathcal{F} is an I -family.

(ii) $x > \bar{x}$. By the stepping-up lemma,

$$|\mathcal{F}_m| \geq \left(1 - (1 - \alpha)\binom{m}{r}\right) \cdot \binom{x}{m},$$

and by Theorem B, \mathcal{F} has an I -subfamily of size ωx where

$$\begin{aligned} \omega &> \left(\left(1 - (1 - \alpha)\binom{m}{r}\right)/\binom{m}{n}\right)^{1/(m-n)} \\ &= \left(1 - (1 - \alpha)\binom{m}{r}\right)^{1/(m-n)}\binom{m}{n}^{1/(n-m)} \\ &> \left(1 - \frac{1 - \rho}{2}\right)^{1/(m-n)}\left(1 - \frac{1 - \rho}{2}\right) = \left(\frac{1 + \rho}{2}\right)^{1+1/(m-n)} > \rho. \end{aligned}$$

This completes the proof of the theorem. \square

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