

## WHITNEY'S TRICK FOR THREE 2-DIMENSIONAL HOMOLOGY CLASSES OF 4-MANIFOLDS

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**ABSTRACT.** In his recent paper, Y. Matsumoto has defined a triple product of 2-homology classes of simply-connected oriented 4-manifolds, when the intersection numbers are zero. In the present paper, the author establishes that three 2-homology classes can be homotopically separated if the intersection numbers and the triple product vanish.

**1. Introduction.** Let  $M$  be a simply-connected oriented 4-manifold possibly with boundary. We shall say that homology classes  $x_i \in H_2(M; \mathbb{Z})$ ,  $i = 1, \dots, n$ , can be separated, if there exist continuous maps  $f_i: S^2 \rightarrow M$  representing  $x_i$ ,  $i = 1, \dots, n$ , such that  $f_i(S^2) \cap f_j(S^2) = \emptyset$  for  $i \neq j$ .

In [2], establishing a homotopy version of Whitney's trick in dimension 4, Kobayashi proved that homology classes  $x_1$  and  $x_2 \in H_2(M; \mathbb{Z})$  can be separated if and only if the intersection number  $x_1 \cdot x_2 = 0$ .

On the other hand, Matsumoto [3] defined a "secondary intersection triple", which we shall call Matsumoto triple,  $\langle x_1, x_2, x_3 \rangle \in \mathbb{Z}/I$ , where  $x_1, x_2, x_3$  are 2-dimensional homology classes of  $M$  such that  $x_i \cdot x_j = 0$  for  $i \neq j$  and  $I$  is an ideal of  $\mathbb{Z}$ ,  $\{x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 \in \mathbb{Z}; y_1, y_2, y_3 \in H_2(M, \mathbb{Z})\}$ . See §4 for the definition. He showed that  $x_1, x_2, x_3$  cannot be separated if the triple  $\langle x_1, x_2, x_3 \rangle \neq 0$ .

In this paper we shall prove the following:

**THEOREM.** *Let  $M$  be a simply-connected oriented 4-manifold, then three homology classes  $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$  can be separated if and only if the intersection numbers  $x_i \cdot x_j = 0$  for  $i \neq j$  and the Matsumoto triple  $\langle x_1, x_2, x_3 \rangle = 0$ .*

**COROLLARY.** *When  $M$  is closed, homology classes  $x_1, x_2, x_3$  can be separated if and only if the intersection numbers  $x_i \cdot x_j = 0$  for  $i \neq j$ .*

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**2. Some fundamental devices.** Throughout this paper, we shall denote by  $M$  a simply-connected oriented 4-manifold. Let  $f_1, f_2: S^2 \rightarrow M$  be smooth generic immersions in the sense that all the self- and mutual-intersections of

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$S_1 (= f_1(S^2))$  and  $S_2 (= f_2(S^2))$  are transversal double points. Suppose that  $S_1 \cap S_2$  consists of positive double points, say  $p_1, \dots, p_m$ , and negative ones, say  $q_1, \dots, q_n$ , where  $m, n \neq 0$ . Draw smooth arcs  $\gamma_1, \gamma_2$  connecting  $p_1$  and  $q_1$  on  $S_1, S_2$ , respectively. We may assume that  $\gamma_1$  and  $\gamma_2$  are generic in the sense that they have no self-intersection points and pass through neither the self-intersection points nor the mutual intersection points of  $S_1$  and  $S_2$ . As  $M$  is simply-connected,  $\gamma_1 \cup \gamma_2$  bounds a smoothly immersed 2-disk  $\Delta$ , called a Whitney disk, which is generic with respect to  $S_1$  and  $S_2$ . Let  $\phi$  be a nonzero vector field on  $\gamma_1 \cup \gamma_2$  such that when restricted to  $\gamma_2$  it gives a cross-section of a normal 1-vector bundle  $\nu(\gamma_2 \hookrightarrow S_2)$  and when restricted to  $\gamma_1$  the unique extension of  $\phi|_{\{p_1, q_1\}}$  over  $\gamma_1$  which is normal to both  $S_1$  and  $\Delta$ . Let  $\theta(\Delta) \in Z = \pi_1(SO_2)$  be the obstruction to extending  $\phi$  over  $\Delta$ . We shall say that the Whitney disk  $\Delta$  is good, if  $\theta(\Delta) = 0$ .

*Device 1 (Making the Whitney disk  $\Delta$  good).* When the Whitney disk  $\Delta$  is not good, one can obtain a good Whitney disk spanning  $\gamma_1 \cup \gamma_2$  by spinning  $\Delta$  around  $\gamma_1$  or  $\gamma_2$  (see [1]).

*Device 2 (Making  $S_1$  escape from the intersection with  $\text{int } \Delta$  across  $\gamma_1$ ).* If  $S_1 \cap \text{int } \Delta \neq \emptyset$ , for a point  $p \in S_1 \cap \text{int } \Delta$  we take a point  $p' \in \text{int } \gamma_1$  and a simple arc  $\gamma$  connecting  $p$  and  $p'$  on  $\Delta$  such that  $\gamma \cap S_1 = \{p, p'\}$  and  $\gamma \cap S_2 = \emptyset$ . Pushing a neighborhood of the intersecting point  $p$  in  $S_1$  along the arc  $\gamma$  off  $\Delta$  as in Figure 1, we can make  $S_1$  escape from the intersection point  $p$  with  $\text{int } \Delta$  across  $\gamma_1$ , by adding two self-intersection points with opposite sign for  $S_1$ .

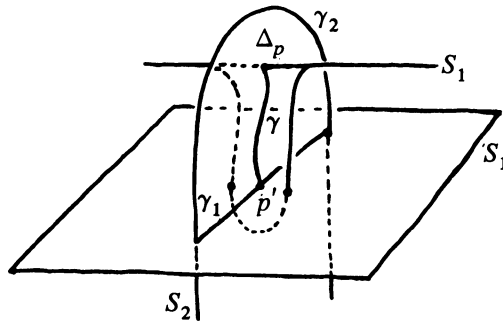


FIGURE 1

*Device 3 (Whitney's trick for  $S_1$  and  $S_2$  across  $\gamma_1$ ).* Let  $\Delta$  be a good immersed Whitney disk such that  $S_1 \cap \text{int } \Delta = \emptyset$ . We set

$$\begin{aligned}
 D &= \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1, y > 0\}, \\
 D' &= \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < (6/5)^2, y > 0\}, \\
 C_1 &= \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1, y > 0\}, \\
 C'_1 &= \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = (6/5)^2, y > 0\},
 \end{aligned}$$

$$C_2 = \{(x, 0) \in R^2; -1 < x < 1\},$$

$$C'_2 = \{(x, 0) \in R^2; -6/5 < x < 6/5\}.$$

Let  $f: D \rightarrow M$  be an immersion such that  $f(D) = \Delta, f(C_1) = \gamma_1, f(C_2) = \gamma_2$ . Adding a collar along  $\gamma_1$ , we have an extension  $f': D' \rightarrow M$  of  $f$ , so that  $f'(C'_1) \cap S_1 = \emptyset, f'(C'_2) \subset S_2$ . Since  $\Delta$  is good, we have an immersion  $\tilde{f}: D' \times [-1, 1] \rightarrow M$  using the vector field  $\tilde{\phi}$  (the extension of  $\phi$  over  $\Delta$ ) such that  $\tilde{f}$  restricted to  $D' = D' \times 0$  coincides with  $f'$ . Now the immersed 2-sphere  $S_2$  shall be modified as follows.

$$S'_2 = (S_2 - \tilde{f}(C'_2 \times [-1, 1])) \cup \tilde{f}(C'_1 \times [-1, 1]) \cup \tilde{f}(D' \times \{-1, 1\}).$$

Rounding the corners, one can assume  $S'_2$  is a generic immersed 2-sphere. Now  $S_1 \cap S'_2 = \{p_2, \dots, p_m, q_2, \dots, q_n\}$ . It is easy to construct a generic immersion  $f'_2: S^2 \rightarrow M$  with  $f'_2(S^2) = S'_2$  which is regularly homotopic to  $f_2$ . This process will be referred to as Whitney's trick for  $S_1$  and  $S_2$  across  $\gamma_1$  (see [4, Theorem 6.6]).

REMARK. (1) Let  $X$  be a compact subset of  $M$ . If  $\Delta \cap X = \emptyset$ , then  $S'_2 \cap X = S_2 \cap X$  in Device 3.

(2) These can be applied to generic intersections of immersed disks and spheres not only of immersed spheres.

Using the Devices 1, 2 and 3 repeatedly, we obtain

PROPOSITION (KOBAYASHI [2]). Let  $x_1, x_2 \in H_2(M; Z)$  be homology classes such that  $x_1 \cdot x_2 = r$ . Then  $x_1$  and  $x_2$  can be represented by continuous maps of  $S^2$  whose images have  $|r|$  points in common. In particular, if  $x_1 \cdot x_2 = 0$ ,  $x_1$  and  $x_2$  can be separated.

3. The key lemma. Let  $S_1, S_2, S_3$  be smoothly immersed generic 2-spheres in  $M$  such that their mutual algebraic intersection numbers are all zero. We denote by  $p_\lambda^{(i,j)}$  (or  $q_\lambda^{(i,j)}$ ) the  $\lambda$ th positive (or negative) intersection point of  $S_i$  and  $S_j$ . Draw a smoothly imbedded arc  $\gamma_{\lambda,i}^{(i,j)}$  (or  $\gamma_{\lambda,j}^{(i,j)}$ ) connecting  $p_\lambda^{(i,j)}$  and  $q_\lambda^{(i,j)}$  on the immersed sphere  $S_i$  (or  $S_j$ ). We assume that  $\gamma_{\lambda,i}^{(i,j)} \cap \gamma_{\mu,i}^{(i,k)} = \emptyset$  ( $\lambda \neq \mu$  or  $j \neq k$ ). Let  $\Delta_\lambda^{(i,j)}$  be a smoothly immersed generic 2-disk bounding the circle  $\gamma_{\lambda,i}^{(i,j)} \cup \gamma_{\lambda,j}^{(i,j)}$ .

LEMMA. Suppose that  $\Delta_1^{(i,j)} \cap S_k = \{a_1, \dots, a_m\}$  and  $\Delta_2^{(i,j)} \cap S_k = \{b_1, \dots, b_n\}$  where  $\{i, j, k\} = \{1, 2, 3\}$ . Then one can regularly homotope  $S_1, S_2, S_3$  to obtain  $S'_1, S'_2, S'_3$  and Whitney disks  $\{\Delta'_\lambda^{(i,j)}\}$  such that:

- (1)  $S'_i \cap S'_j = S_i \cap S_j$  ( $\forall i, j$ ),
- (2)  $\Delta'_\lambda^{(i,j)} \cap S'_k = \Delta_\lambda^{(i,j)} \cap S_k$ , for  $\lambda > 2$ , and
- (3)  $\Delta'_1^{(i,j)} \cap S'_k = \{a_2, \dots, a_m\}, \Delta'_2^{(i,j)} \cap S'_k = \{b_0, b_1, \dots, b_n\}$  where  $b_0$  and  $a_1$  have the same sign.

PROOF. (See Figure 2.) Make  $S_k$  escape from the intersection point  $a_1$  with  $\Delta_1^{(i,j)}$  across  $\gamma_{1,i}^{(i,j)}$ , adding new intersections of  $S_k$  and  $S_i, p_0^{(k,i)}$  and  $q_0^{(k,i)}$ ; then we obtain a small Whitney disk  $\Delta'$ . Choose an imbedding  $g: B = [-1, 1] \times [0, 1] \rightarrow S_i$  such that

$$g([-1, 1] \times \{0\}) = \partial\Delta' \cap S_i, \quad g(\{1\} \times \{0\}) = p_0^{(k,i)},$$

$$g(\{-1\} \times \{0\}) = q_0^{(k,i)}, \quad g(B) \cap \gamma_{\lambda,i}^{(i,j)} = \emptyset$$

except for  $l = j$  and  $\lambda = 1$  or  $2$ , and

$$g(B) \cap \gamma_{1,i}^{(i,j)} = \gamma_{1,i}^{(i,j)}\left(\left[0, \frac{1}{2}\right]\right) = g_1, \quad g(B) \cap \gamma_{2,i}^{(i,j)} = \gamma_{2,i}^{(i,j)}\left(\left[\frac{1}{2}, 1\right]\right) = g_2,$$

where

$$\gamma_{1,i}^{(i,j)}(0) = p_1^{(i,j)}, \quad \gamma_{2,i}^{(i,j)}(1) = q_2^{(i,j)}.$$

Let  $\gamma_{0,k}^{(k,i)} = \partial\Delta' \cap S_k$ ,

$$\gamma_{0,i}^{(k,i)} = g(\{1, -1\} \times [0, 1] \cup [-1, 1] \times \{1\}),$$

and  $\tilde{\Delta}_0^{(k,i)} = g(B) \cup \Delta'$ . Then  $\tilde{\Delta}_0^{(k,i)} \cap S_j = \{p_1^{(i,j)}, q_2^{(i,j)}\}$ . Let  $\psi$  be a vector field on the arc  $g(\{0\} \times [0, 1])$  which does not lie in  $T(S_i)$ ,  $T(S_i)|_{g_1} + T(\Delta_l^{(i,j)})|_{g_l}$  ( $l = 1, 2$ ). Push  $\tilde{\Delta}_0^{(k,i)}$  off  $S_i$  along  $\psi$  keeping  $\gamma_{0,i}^{(k,i)}$  fixed; then we obtain an imbedded disk  $\tilde{\Delta}_0^{(k,i)}$  bounded by  $\gamma_0^{(k,i)} \cup \gamma_{0,k}^{(k,i)}$  such that it meets  $S_i$  normally along  $\gamma_{0,i}^{(k,i)}$ ,  $S_k$  normally along  $\gamma_{0,k}^{(k,i)}$ , and

$$S_i \cap \text{int } \tilde{\Delta}_0^{(k,i)} = \emptyset, \quad S_k \cap \text{int } \tilde{\Delta}_0^{(k,i)} = \emptyset,$$

$$S_j \cap \tilde{\Delta}_0^{(k,i)} = \{p, q\}, \quad \tilde{\Delta}_0^{(k,i)} \cap \Delta_\lambda^{(i,j)} = \emptyset$$

for  $\lambda = 1, 2$ . We can cancel these intersection points  $p$  and  $q$  as follows. Let  $\gamma$  (or  $\gamma'$ ) be a generic arc connecting  $p$  and  $q$  on  $\tilde{\Delta}_0^{(k,i)}$  (or  $S_j$ ), and let  $\Delta$  be a good generic immersed disk bounded by  $\gamma \cup \gamma'$ . We can make  $S_j$  escape from the intersection with  $\text{int } \Delta$  across  $\gamma'$ . Doing Whitney's trick for  $\tilde{\Delta}_0^{(k,i)}$  and  $S_j$  across  $\gamma'$ , we obtain a new immersed disk  $\Delta_0^{(k,i)}$  such that  $S_j \cap \Delta_0^{(k,i)} = \emptyset$ . We may assume that  $\Delta_0^{(k,i)}$  is good, and

$$\Delta_0^{(k,i)} \cap \Delta_\lambda^{(j,k)} = \emptyset, \quad \text{int } \Delta_0^{(k,i)} \cap \text{int } \Delta_\mu^{(i,j)} = \emptyset.$$

For example, if  $\Delta_0^{(k,i)} \cap \Delta_\lambda^{(j,k)} \neq \emptyset$ , we can make  $\Delta_0^{(k,i)}$  escape from the intersection with  $\Delta_\lambda^{(j,k)}$  across  $\gamma_{\lambda,k}^{(j,k)}$  by adding two intersection points of  $\Delta_0^{(k,i)}$  and  $S_k$ . Possibly  $\text{int } \Delta_0^{(k,i)} \cap S_k \neq \emptyset$ ,  $\text{int } \Delta_0^{(k,i)} \cap S_i \neq \emptyset$ . Make  $\Delta_0^{(k,i)}$  escape from this intersection with  $S_k$  across  $\gamma_{0,k}^{(k,i)}$ , if necessary.

Using  $\Delta_0^{(k,i)}$ , we can do Whitney's trick for  $S_k$  and  $S_i$  across  $\gamma_{0,k}^{(k,i)}$  and we obtain a new immersed 2-sphere  $S'_i$  such that

$$S'_i \cap S_k = S_i \cap S_k - \{p_0^{(k,i)}, q_0^{(k,i)}\} \quad \text{and} \quad \Delta_\lambda^{(j,k)} \cap S'_i = \Delta_\lambda^{(j,k)} \cap S_i.$$

Let  $\tilde{f}$  denote the immersion:  $D' \times [-1, 1] \rightarrow M$  in Device 3 such that  $\tilde{f}(C_2 \times \{0\}) = \gamma_{0,i}^{(k,i)}$ . We may assume that

$$\tilde{f}(\{(0, 0)\} \times [-1, 1]) = \gamma_{2,i}^{(i,j)} \cap \tilde{f}(D' \times [-1, 1]).$$

We shall modify the disk  $\Delta_2^{(i,j)}$  as follows:

$$\Delta_2^{(i,j)} = \Delta_2^{(i,j)} \cup \tilde{f}(\{(x, 0); 0 < x < 6/5\} \times [-1, 1]).$$

Then we obtain a new intersection point  $b_0$ . Q.E.D.

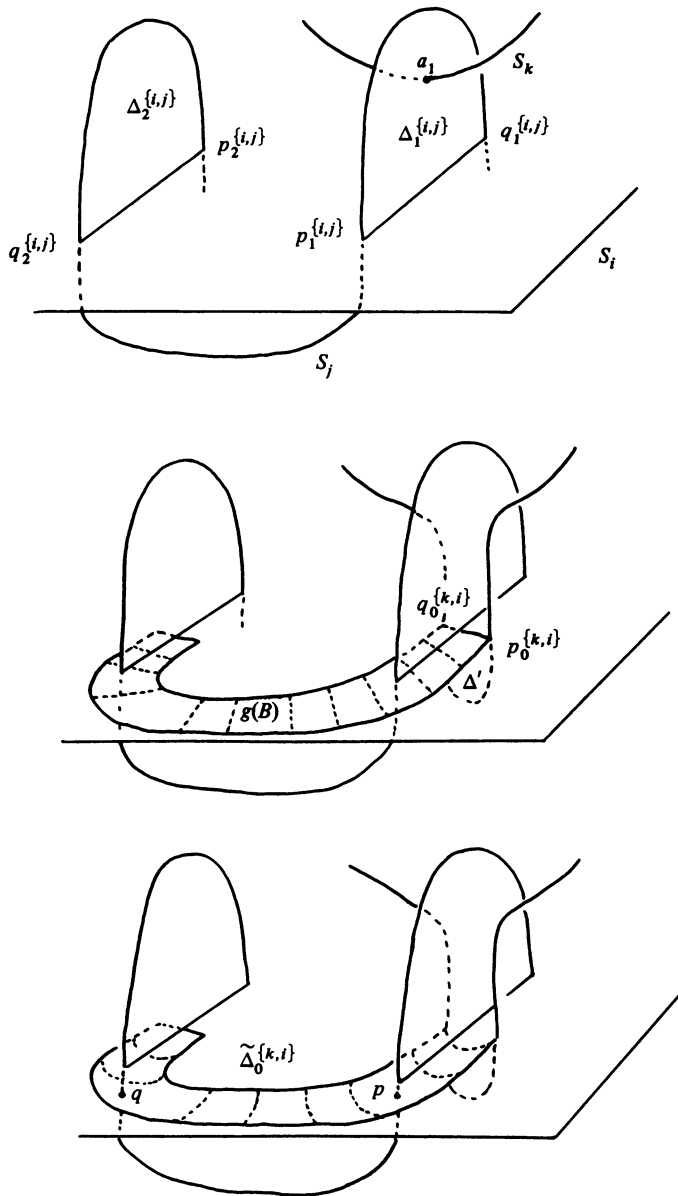


FIGURE 2

**4. Proof of theorem and corollary.** Let  $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$  be homology classes such that  $x_i \cdot x_j = 0$  for  $i \neq j$ . Represent  $x_1, x_2, x_3$  by smoothly immersed generic 2-spheres  $S_1, S_2, S_3$ , and let  $p_\lambda^{(i,j)}, \gamma_{\lambda,i}^{(i,j)}, \Delta_\lambda^{(i,j)}$  be as in §3, but we do not require the condition  $\gamma_{\lambda,i}^{(i,j)} \cap \gamma_{\mu,i}^{(i,k)} = \emptyset$  ( $\lambda \neq \mu$  or  $j \neq k$ ). The Whitney disk  $\Delta_\lambda^{(i,j)}$  is oriented as in Figure 3. Now the Matsumoto triple  $\langle x_1, x_2, x_3 \rangle$  is defined as follows:

$$\begin{aligned} \langle x_1, x_2, x_3 \rangle &= \sum_{\lambda} S_1 \cdot \Delta_{\lambda}^{\{2,3\}} + \sum_{\mu} S_2 \cdot \Delta_{\mu}^{\{3,1\}} + \sum_{\nu} S_3 \cdot \Delta_{\nu}^{\{1,2\}} \\ &+ \sum_{\mu, \nu} \frac{\partial \Delta_{\mu}^{\{3,1\}} \cdot \partial \Delta_{\nu}^{\{1,2\}}}{S_1} + \sum_{\nu, \lambda} \frac{\partial \Delta_{\nu}^{\{1,2\}} \cdot \partial \Delta_{\lambda}^{\{2,3\}}}{S_2} \\ &+ \sum_{\lambda, \mu} \frac{\partial \Delta_{\lambda}^{\{2,3\}} \cdot \partial \Delta_{\mu}^{\{3,1\}}}{S_3} \pmod I, \end{aligned}$$

where  $S_1 \cdot \Delta_{\lambda}^{\{2,3\}}$ , etc., denote the intersection number of  $S_1$  and  $\Delta_{\lambda}^{\{2,3\}}$ , etc., and  $(\partial \Delta_{\mu}^{\{3,1\}} \cdot \partial \Delta_{\nu}^{\{1,2\}})/S_1$ , etc. denote the intersection number of  $\partial \Delta_{\mu}^{\{3,1\}}$  and  $\partial \Delta_{\nu}^{\{1,2\}}$  on  $S_1$ , etc.

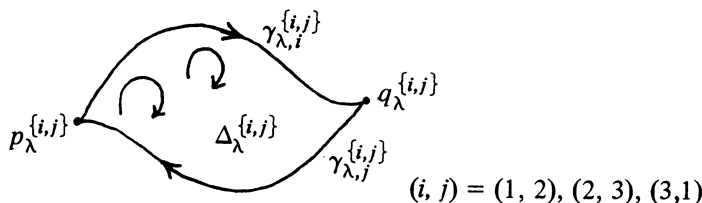


FIGURE 3

**PROOF OF THEOREM.** Let  $x_1, x_2, x_3 \in H_2(M; Z)$  be homology classes such that  $x_i \cdot x_j = 0$  for  $i \neq j$  and  $\langle x_1, x_2, x_3 \rangle = 0 \in Z/I$ . Let  $S_i, \gamma_{\lambda,i}^{\{i,j\}}, \Delta_{\lambda}^{\{i,j\}}$  be as above. Now we assume as in §3 that  $\gamma_{\lambda,i}^{\{i,j\}} \cap \gamma_{\mu,i}^{\{i,k\}} = \emptyset$  ( $\lambda \neq \mu$  or  $j \neq k$ ); then the Matsumoto triple  $\langle x_1, x_2, x_3 \rangle$  is defined as

$$\sum_{\lambda} S_1 \cdot \Delta_{\lambda}^{\{2,3\}} + \sum_{\mu} S_2 \cdot \Delta_{\mu}^{\{3,1\}} + \sum_{\nu} S_3 \cdot \Delta_{\nu}^{\{1,2\}}.$$

We may assume that this sum is zero. In fact, if the ideal  $I$  is  $\{0\}$ , it is always zero. If  $I$  is not  $\{0\}$ , there exist homology classes  $y_1, y_2, y_3 \in H_2(M; Z)$  such that  $\langle x_1, x_2, x_3 \rangle = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$ . Let  $F_1, F_2, F_3$  be immersed 2-spheres representing  $y_1, y_2, y_3$ . Make connected-sums of  $\Delta_{\lambda}^{\{2,3\}}$  and  $-F_1, \Delta_{\lambda}^{\{3,1\}}$  and  $-F_2, \Delta_{\lambda}^{\{1,2\}}$  and  $-F_3$ , where  $-F$  is an immersed 2-sphere with the reversed orientation. Then if we use the resulting immersed disks instead of  $\Delta_{\lambda}^{\{2,3\}}, \Delta_{\lambda}^{\{3,1\}}, \Delta_{\lambda}^{\{1,2\}}$ , the sum is zero. We may assume that every Whitney disk is good in the sense of §2 and that there is no mutual-intersection of Whitney disks (and even there is no self-intersection of Whitney disks, i.e. every Whitney disk is an imbedded disk). (See proof of lemma.) As  $x_1 \cdot x_3 = 0$ , we may assume  $S_1 \cap S_3 = \emptyset$  by the proposition. Escaping the intersection  $S_1 \cap \text{int } \Delta_{\lambda}^{\{1,2\}}$  across  $\gamma_{\lambda,i}^{\{1,2\}}$ , we obtain a new immersed 2-sphere  $S'_1$ , so that  $S'_1 \cap \text{int } \Delta_{\lambda}^{\{1,2\}} = \emptyset$  and  $S'_1 \cap S_3 = \emptyset$ . Using  $\Delta_{\lambda}^{\{1,2\}}$ , do Whitney's trick for  $S'_1$  and  $S_2$  across  $\gamma_{\lambda,i}^{\{1,2\}}$ , and we shall obtain a new immersed 2-sphere  $S'_2$  such that  $S'_1 \cap S'_2 = \emptyset$ . By Lemma, we may assume that  $S'_1 \cap \Delta_{\lambda}^{\{2,3\}} = \emptyset$  for  $\lambda \neq 1$ . Then  $S'_1 \cdot \Delta_{\lambda}^{\{2,3\}} = 0$ . Using Devices 1, 2 and 3, we obtain  $S'_1 \cap \Delta^{\{2,3\}} = \emptyset$ . Now we can do Whitney's trick for  $S'_2$  and  $S_3$  (using Device 2), and we obtain the required maps. Q.E.D.

PROOF OF COROLLARY. If one of  $x_1, x_2, x_3$  is 0, then this follows immediately from the proposition. If one of  $x_1, x_2, x_3$ , say  $x_1$ , is a primitive element, i.e. there is no homology class  $x \in H_2(M; Z)$  such that  $x_1 = mx$  ( $m \in Z, \neq 1, -1$ ), then there exists a homology class  $y \in H_2(M; Z)$  such that  $x_1 \cdot y = 1$ . Therefore  $\langle x_1, x_2, x_3 \rangle = 0 \pmod{I = (1)}$  and they can be separated by the theorem. If  $x_1, x_2, x_3$  can be separated, also  $mx_1, x_2, x_3$  can be separated by using the "self-connected-sum" of the immersed 2-sphere representing  $x_i$  ( $m \in Z$ ). Q.E.D.

## REFERENCES

1. M. Freedman and R. Kirby, *A geometric proof of Rochlin's theorem*, Proc. Sympos. Pure Math., vol. 32, Part 2, Amer. Math. Soc., Providence, R. I., 1978, pp. 85–98.
2. K. Kobayashi, *On a homotopy version of 4-dimensional Whitney's lemma*, Math. Seminar Notes Kobe Univ. 5 (1977), 109–116.
3. Y. Matsumoto, *Secondary intersectional properties of 4-manifolds and Whitney's trick*, Proc. Sympos. Pure Math., vol. 32, Part 2, Amer. Math. Soc., Providence, R. I., 1978, pp. 99–107.
4. J. Milnor, *Lectures on the h-cobordism theorem*, Math. Notes, vol. 1, Princeton Univ. Press, Princeton, N. J., 1965.

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