PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 1, August 1979

THE NOETHERIAN PROPERTY FOR QUOTIENT RINGS OF INFINITE POLYNOMIAL RINGS

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ABSTRACT. Let X be an infinite set of indeterminates over the commutative Noetherian ring R with identity. We prove that the quotient ring of $R[X]$ with respect to the multiplicative system of polynomials of unit content is also Noetherian. Moreover, we show that certain quotient rings of $R[X]$ with respect to multiplicative systems of monic polynomials (where "monic" is appropriately defined) are also Noetherian.

Let $X = \{X_\lambda\}_{\lambda \in \Lambda}$ be a set of indeterminates over R, a commutative ring with identity. The *content* of a polynomial $f \in R[X]$ is denoted by $C(f)$ and is defined to be the ideal of R generated by the coefficients of f; if $C(f) = R$, then f has *unit content*. The quotient ring of $R[X]$ with respect to the multiplicative set of polynomials of unit content is denoted by $R(X)$. Ray Heitmann has asked[3] if $R(X)$ is Noetherian, provided the ring R is Noetherian. We show in Theorem 6 that the answer to this question is affirmative. We subsequently define other quotient rings of $R[X]$, contained in $R(X)$ and defined in terms of certain multiplicative sets of "monic" polynomials of $R[X]$. We show the Noetherian property to be preserved also for these quotient rings of $R[X]$.

The ring $R(X)$ seems to have been first considered by Krull in **[5]**, while Nagata introduced the notation $R(X)$ in **[10**, p. 17]. The following result, which is a partial citation of Proposition 33.1 of **[2]**, lists some basic properties of the ring $R(X)$.

PROPOSITION 1. *Let S be the set of elements of $R[X]$ of unit content, and let $\{M_\beta\}_{\beta \in B}$ be the set of maximal ideals of R. For each $\beta \in B$, denote by $M_\beta[X]$ and $M_\beta(X)$ the ideals of $R[X]$ and $R(X)$, respectively, generated by M_β.*

(1) $S = R[X] - (\bigcup_{\beta \in B} M_\beta[X])$.

(2) *$\{M_\beta[X]\}$ is the family of ideals of $R[X]$ maximal with respect to the property of failure to meet the multiplicative system S; hence $\{M_\beta(X)\}$ is the set of maximal ideals of $R(X)$.*

(3) *If Q is an ideal of R, then $QR(X) \cap R = Q$; if Q is P-primary in R, then $QR(X)$ is $PR(X)$-primary.*

Presented to the Society November 12, 1977; received by the editors September 9, 1978.
AMS (MOS) subject classifications (1970). Primary 13B25, 13E05; Secondary 13C15.
Key words and phrases. Noetherian ring, polynomial of unit content, monic polynomial.
[1]Research supported by NSF Grant MCS 75-06591.
[2]Research supported by NSF Grant 7800798.
[3]Private communication.

For a prime ideal P of a commutative ring, we use $h(P)$ to denote the height of P ([**2**, p. 109], [**10**, p. 24]).

LEMMA 2. *If R is Noetherian, if T is a quotient ring of the polynomial ring $R[\{X_\lambda\}]$, and P is a prime ideal of T, then $h(P) \geqslant h(P \cap R)$, and equality holds if and only if $P = (P \cap R)T$.*[4]

PROOF. Since prime ideals of T are the extensions of their contractions in $R[\{X_\lambda\}]$, it suffices to prove the result for $T = R[\{X_\lambda\}]$. Moreover, since prime ideals extend to prime ideals and contract to themselves with respect to polynomial ring extension, and since $R[\{X_\lambda\}]$ is the direct union of the polynomial rings over R on finite subsets of $\{X_\lambda\}$, it suffices to prove the result for $T = R[X_1, \ldots, X_n]$, and hence just for the case of a polynomial ring in one variable over a Noetherian ring. The result in this case is well known ([**6**, §6], [**4**, Theorem 149]).

LEMMA 3. *If R is Noetherian, if T is a quotient ring of the polynomial ring $R[\{X_\lambda\}]$, and P is a prime ideal of T of finite height, then there exists a finite subset $\{X_i\}_{i=1}^n$ of $\{X_\lambda\}$ such that $P = (P \cap R[X_1, \ldots, X_n])T$.*

PROOF. Again, since each prime ideal of T is the extension of its contraction to $R[\{X_\lambda\}]$, we may assume that $T = R[\{X_\lambda\}]$. If $h(P) = r$, let $P_0 < P_1 < \cdots < P_r = P$ be a chain of prime ideals of T, and let $f_i \in P_i \setminus P_{i-1}$, $i = 1, \ldots, r$. The f_i are polynomials in a finite subset $\{X_i\}_{i=1}^n$ of $\{X_\lambda\}$ and $h(P) = h(P \cap R[X_1, \ldots, X_n])$. Hence, $P = (P \cap R[X_1, \ldots, X_n])T$ by Lemma 2.

THEOREM 4. *If R is Noetherian, if T is a quotient ring of the polynomial ring $R[\{X_\lambda\}]$, and P is a prime ideal of T of finite height, then P is finitely generated.*

PROOF. By Lemma 3, $P = (P \cap R[X_1, \ldots, X_n])T$. Since R is Noetherian, $R[X_1, \ldots, X_n]$ is Noetherian by the Hilbert Basis Theorem. Hence $P \cap R[X_1, \ldots, X_n]$, and therefore P, is finitely generated.

THEOREM 5. *If R is Noetherian and T is a quotient ring of the polynomial ring $R[\{X_\lambda\}]$, then T is Noetherian if and only if each prime ideal of T has finite height.*

PROOF. By Krull's Principal Ideal Theorem, each prime ideal of a Noetherian ring has finite height. Therefore the condition is clearly necessary. If each prime ideal of T has finite height, then, by Theorem 4, each prime ideal of T is finitely generated, so by a theorem of Cohen ([**4**, p. 5] or [**10**, p. 8]) T is Noetherian.

[4]If R is not Noetherian, it can happen that $P = (P \cap R)R[X]$ and $h(P) > h(P \cap R)$ [**2**, p. 364].

THEOREM 6. *If R is Noetherian, then $R(X) = R(\{X_\lambda\}_{\lambda\in\Lambda})$ is also Noetherian.*

PROOF. By Theorem 5 and Proposition 1, it suffices to show for each maximal ideal M of R that $MR[\{X_\lambda\}]$ has finite height. Since R is Noetherian, M is of finite height, and, by Lemma 2, $h(M) = h(MR[\{X_\lambda\}])$.

Assume that Y is an indeterminate over the ring R. A ring closely related to $R(Y)$ is the quotient ring of $R[Y]$ with respect to the multiplicative system of monic polynomials over R. We denote this ring by $R\langle Y\rangle$; it arises in Quillen's proof of the Serre Conjecture [**11**],[5] and has been the object of some other recent investigation ([**1**], [**7**, Chapter IV], [**8**]). For a polynomial ring in two variables over R, say $R[X_1, X_2]$, a natural analogue of $R\langle Y\rangle$ is obtained by defining $R\langle X_1, X_2\rangle$ to be the quotient ring of $R\langle X_1\rangle[X_2]$ at the multiplicative system of monic polynomials in X_2 over $R\langle X_1\rangle$. By induction we define

$$R\langle X_1, \ldots, X_n\rangle = R\langle X_1, \ldots, X_{n-1}\rangle\langle X_n\rangle.$$

Note that the definition of $R\langle X_1, \ldots, X_n\rangle$, unlike that of $R(X_1, \ldots, X_n)$, depends upon the order of the indeterminates. For example, it is proved in [**3**, Proposition 10] that $R\langle X_1, X_2\rangle$ and $R\langle X_2, X_1\rangle$ are equal as subrings of the total quotient ring of $R[X_1, X_2]$ if and only if R is 0-dimensional.

In general, if Λ is a totally ordered set and $\{X_\lambda\}_{\lambda\in\Lambda}$ is a set of indeterminates over R, then we define $R\langle\{X_\lambda\}\rangle$ to be the union of the directed set $\{R\langle X_{\lambda_1}, \ldots, X_{\lambda_n}\rangle\}$ of subrings of the total quotient ring of $R[\{X_\lambda\}]$, the union being taken over all finite subsets $\{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ of Λ. It is clear that $R\langle\{X_\lambda\}\rangle$ so defined is a regular quotient ring[6] of $R[\{X_\lambda\}]$. We proceed to show that $R\langle\{X_\lambda\}\rangle = R[\{X_\lambda\}]_S$, where S is the multiplicative system of "monic" polynomials in $R[\{X_\lambda\}]$, defined in the following manner. The total order on Λ induces, via the reverse lexicographic order, a total order on the set of monomials[7] in the indeterminates X_λ, as follows. If $\lambda_1, \ldots, \lambda_n \in \Lambda$ are such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and if $e_1, \ldots, e_n$, $f_1, \ldots, f_n$ are nonnegative integers, then

$$X_{\lambda_1}^{e_1}X_{\lambda_2}^{e_2}\cdots X_{\lambda_n}^{e_n} < X_{\lambda_1}^{f_1}X_{\lambda_2}^{f_2}\cdots X_{\lambda_n}^{f_n}$$

if and only if $e_i < f_i$ for the largest integer i such that the corresponding exponents are unequal. This order on monomials is compatible with multiplication. If g is a nonzero element of $R[\{X_\lambda\}]$, then g can be written in the form

$$g = r_1Y_1 + \cdots + r_kY_k,$$

[5]Quillen uses the notation $R(Y)$ for the quotient ring of $R[Y]$ with respect to the multiplicative system of monic polynomials.

[6]Regularity follows from the fact that an element of $R[X]$ is a zero divisor if and only if it is annihilated by a nonzero element of R ([**8**], [**2**, (28.7)]).

[7]By *monomial*, it is to be understood here that the coefficient from R is 1.

where $r_i \in R$, $r_k \neq 0$, and $Y_1, \ldots, Y_k$ are monomials such that $Y_1 < \cdots < Y_k$. We call Y_k the *leading monomial* of g, r_k the *leading coefficient* of g, and say that g is *monic* if $r_k = 1$. Let S be the set of monic polynomials in $R[\{X_\lambda\}]$. It is clear that S is a multiplicative system in $R[\{X_\lambda\}]$. Since $S \cap R[X_{\lambda_1}, \ldots, X_{\lambda_n}]$ is the set of monic polynomials in $R[X_{\lambda_1}, \ldots, X_{\lambda_n}]$, and since $R\langle\{X_\lambda\}\rangle$ is defined to be $\cup \{R\langle X_{\lambda_1}, \ldots, X_{\lambda_n}\rangle | \{\lambda_i\}_1^n$ is a finite subset of $\Lambda\}$, it is sufficient to establish the equality $R\langle\{X_\lambda\}\rangle = R[\{X_\lambda\}]_S$ in the case where Λ is finite. For the case of one variable, say X_1, $R\langle X_1\rangle = R[X_1]_S$ by definition. For n variables, say $X_1, \ldots, X_n$, it is clear that $R[X_1, \ldots, X_n]_S$ is contained in $R\langle X_1, \ldots, X_n\rangle$, and proceeding by induction, we may assume that $R\langle X_1, \ldots, X_{n-1}\rangle[X_n]$ is contained in $R[X_1, \ldots, X_n]_S$. If f is a polynomial in $R\langle X_1, \ldots, X_{n-1}\rangle[X_n]$ that is monic as a polynomial in X_n with coefficients in $R\langle X_1, \ldots, X_{n-1}\rangle$, then we wish to show that f is a unit in $R[X_1, \ldots, X_n]_S$. We have $f = X_n^m + a_{m-1}X_n^{m-1} + \cdots + a_0$ with the $a_i \in R\langle X_1, \ldots, X_{n-1}\rangle$. By the induction hypothesis, there exists $s \in S \cap R[X_1, \ldots, X_{n-1}]$ such that $sa_i \in R[X_1, \ldots, X_{n-1}]$ for $i = 1, \ldots, m-1$. If s has leading monomial Y, then YX_n^m is the leading monomial of $sf \in R[X_1, \ldots, X_n]$, and sf is a monic polynomial in $R[X_1, \ldots, X_n]$. Therefore $sf \in S$ and f is a unit of $R[X_1, \ldots, X_n]$. We conclude that $R\langle\{X_\lambda\}\rangle = R[\{X_\lambda\}]_S$.

We shall prove in Theorem 9 that the ring $R\langle\{X_\lambda\}\rangle$ is Noetherian if R is Noetherian; for finite dimensional R, this follows readily from Theorem 5.

PROPOSITION 7. *Let Λ be a totally ordered set and let $\{X_\lambda | \lambda \in \Lambda\}$ be a set of indeterminates over the finite dimensional Noetherian ring R. Then $R\langle\{X_\lambda\}_{\lambda\in\Lambda}\rangle$ is Noetherian.*

PROOF. By Theorem 5, it suffices to show that $R\langle\{X_\lambda\}\rangle$ is finite dimensional. If J is an n-dimensional Noetherian ring, it is known that $J\langle Y\rangle$ again has dimension n ([**1**], [**7**, Proposition 1.2, Chapter IV]). Since $R\langle\{X_\lambda\}\rangle$ is the direct limit of the rings $R\langle X_{\lambda_1}, \ldots, X_{\lambda_s}\rangle$ for finite subsets $\{X_{\lambda_i}\}_{i=1}^s$ of $\{X_\lambda\}$, and since the direct limit of rings of dimension n is a ring of dimension $\leq n$, we conclude that $R\langle\{X_\lambda\}\rangle$ is Noetherian. (In fact, dim $R\langle\{X_\lambda\}\rangle = n$.)

REMARK 8. If N is a multiplicative system in R, then there exist canonical homomorphisms φ: $R(X)_N \to R_N(X)$ and ψ: $R\langle X\rangle_N \to R_N\langle X\rangle$. In general, these canonical homomorphisms are not surjective. This is the case even if N is the complement of a maximal ideal of R. For example, if R is a polynomial ring in two variables over an algebraically closed field and N is the complement of a maximal ideal of R, then it is easy to see that $R(X)_N$ and $R\langle X\rangle_N$ are properly contained in $R_N(X)$ and $R_N\langle X\rangle$ as subrings of the quotient field of $R[X]$. Thus, to show $R\langle\{X_\lambda\}\rangle$ is Noetherian for R an infinite dimensional Noetherian ring, something other than just a naive localization argument and an application of Proposition 7 is necessary. We show below, however, that a modified localization argument does work.

THEOREM 9. *If Λ is a totally ordered set and $\{X_\lambda | \lambda \in \Lambda\}$ is a set of indeterminates over the Noetherian ring R, then $R\langle\{X_\lambda\}\rangle$ is Noetherian.*

PROOF. Let S be the multiplicative system of monic polynomials in $R[\{X_\lambda\}]$. We have $R[\{X_\lambda\}]_S = R\langle\{X_\lambda\}\rangle$, and to show that $R\langle\{X_\lambda\}\rangle$ is Noetherian, it suffices, by Theorem 5, to show that if P is a prime ideal of $R[\{X_\lambda\}]$ not meeting S, then P is of finite height. Consider the set A consisting of zero and the set of leading coefficients of elements of P. It is clear that A is closed under multiplication by elements of R. We note that A is also closed under subtraction. For suppose r_1 and r_2 are the leading coefficients of f_1 and f_2 in P. Certainly $r_1 - r_2$ is in A if $r_1 = 0$, $r_2 = 0$, or $r_1 = r_2$. Otherwise, if m_i is the leading monomial of f_i, then $r_1 - r_2$ is the leading coefficient of $f_1 m_2 - f_2 m_1 \in P$. Therefore A is an ideal of R, and since P does not meet S, the ideal A is proper. Let M be a maximal ideal of R containing A, and let φ denote the canonical homomorphism of $R[\{X_\lambda\}]$ into $R_M[\{X_\lambda\}]$. We prove that the prime ideal $\varphi(P)R_M[\{X_\lambda\}]$ does not meet the multiplicative system of monic polynomials of $R_M[\{X_\lambda\}]$. Suppose not. Then there exist $f \in P$ and $r \in R \setminus M$ such that $\varphi(f)/\varphi(r)$ is a monic polynomial in $R_M[\{X_\lambda\}]$. If Y is the leading monomial of $\varphi(f)/\varphi(r)$ in $R_M[\{X_\lambda\}]$, then there exists $t \in R \setminus M$ such that tf is in $R[\{X_\lambda\}]$ and has leading monomial Y. If u is the leading coefficient of tf, then $\varphi(u) = \varphi(rt)$. Since $rt \notin M$, we have $u \notin M$. But $tf \in P$ so that $u \in A$. This contradicts the fact that A is contained in M. Therefore $\varphi(P)R_M[\{X_\lambda\}]$ extends to a proper ideal in $R_M\langle\{X_\lambda\}\rangle$. Since R_M is a finite dimensional Noetherian ring, $R_M\langle\{X_\lambda\}\rangle$ is Noetherian by Proposition 7. Therefore $\varphi(P)R_M[\{X_\lambda\}]$ is a prime ideal of finite height. Since $h(P) = h(\varphi(P)R_M[\{X_\lambda\}])$, this completes the proof of Theorem 9.

If σ is a nonidentity permutation on the set of positive integers, and R is a ring of dimension greater than 0, then it follows from [3, Proposition 10] that $R\langle X_1, X_2, \ldots\rangle$ and $R\langle X_{\sigma 1}, X_{\sigma 2}, \ldots\rangle$ are distinct subrings of the total quotient ring of $R[\{X_i\}]$. Of course, these rings are R-isomorphic under the mapping taking X_i to $X_{\sigma i}$, and by Theorem 9, each is Noetherian if R is Noetherian. It seems natural, therefore, to consider the ring $T = \bigcap\{R\langle X_{\sigma 1}, X_{\sigma 2}, \ldots\rangle | \sigma$ is a permutation on the set of positive integers$\}$, and the multiplicative system S of $R[\{X_i\}]$ consisting of polynomials that are monic with respect to each ordering of the set of positive integers defined by a permutation σ. We proceed to show that $T = R[\{X_i\}]_S$, and that this ring is Noetherian if R is Noetherian.

Let G_n denote the permutation group on $\{1, \ldots, n\}$, and for $\sigma \in G_n$, let N_σ denote the multiplicative system of monic polynomials in $R[X_1, \ldots, X_n]$ under the ordering $\sigma 1 < \sigma 2 < \cdots < \sigma n$. Thus $R[X_1, \ldots, X_n]_N = R\langle X_{\sigma 1}, \ldots, X_{\sigma n}\rangle$. Note that any monomial in $X_1, \ldots, X_n$ is in N_σ, and if f is an element of N_σ and m is a monomial, then mf is again in N_σ.

LEMMA 10. *If f_σ is an element of N_σ for each σ in G_n, then there exist monomials m_σ in $X_1, \ldots, X_n$ such that $f = \sum m_\sigma f_\sigma$ is in each N_σ. In particular, if A is an ideal of $R[X_1, \ldots, X_n]$ meeting each of the multiplicative systems N_σ, then A contains an element f such that f is in each N_σ.*

PROOF. We proceed by induction on n, the case $n = 1$ being immediate. Let $\sigma_i \in G_n$ be such that $\sigma_i n = i$ for $i = 1, \ldots, n$. For a fixed i, consider all τ in G_n such that $\tau n = i$. Using the induction hypothesis that the result is true for $n - 1$, we obtain a linear combination with monomial coefficients of the f_τ such that $\tau n = i$, say g_i, that is in every N_τ for which $\tau n = i$. It remains to show that there exist monomials, say p_i, in $X_1, \ldots, X_n$ such that $\sum_{i=1}^n p_i g_i$ is in every N_σ. We choose $p_i = X_i^{e_i}$, where e_i is a positive integer greater than the degree in X_i of any g_j. It follows that $\sum_{i=1}^n X_i^{e_i} g_i$ is in every N_σ.

THEOREM 11. *Let R be a Noetherian ring and let S be the multiplicative system in $R[\{X_i\}_{i=1}^\infty]$ of polynomials that are monic with respect to each ordering of the X_i. Then $R[\{X_i\}]_S$ is again Noetherian, and is equal to $\bigcap \{R\langle X_{\sigma 1}, X_{\sigma 2}, \ldots \rangle | \sigma$ is a permutation on the set of positive integers$\}$.*

PROOF. For each permutation σ, let S_σ denote the multiplicative system of polynomials of $R[\{X_i\}]$ that are monic with respect to the ordering $\sigma 1 < \sigma 2 < \ldots$. Then $S = \bigcap_\sigma S_\sigma$, and to show that $R[\{X_i\}]_S$ is Noetherian, it suffices, by Theorem 5, to show that each prime ideal P of $R[\{X_i\}]$ that does not meet S is of finite height. If P does not meet S_σ for some σ, then it is clear that P is of finite height, for $R[\{X_i\}]_{S_\sigma} = R\langle X_{\sigma 1}, X_{\sigma 2}, \ldots \rangle$ is Noetherian by Theorem 9. Hence $PR\langle X_{\sigma 1}, X_{\sigma 2}, \ldots \rangle$, and therefore P, is of finite height. We show that if P is a prime ideal of $R[\{X_i\}]$ that meets each S_σ, then P meets S. Consider $R[X_1, \ldots, X_n]$ and the $n!$ multiplicative systems N_τ of $R[X_1, \ldots, X_n]$ associated with permutations τ of $\{1, \ldots, n\}$. For each S_σ we have $S_\sigma \cap R[X_1, \ldots, X_n] = N_\tau$ for some τ. If P meets each of the N_τ on $R[X_1, \ldots, X_n]$, then, by Lemma 10, P contains an element f such that f belongs to each N, and therefore to each S_σ, and hence to $S = \bigcap_\sigma S_\sigma$. But, if for each positive integer n, there is a multiplicative system N_{τ_n} on $R[X_1, \ldots, X_n]$ such that $P \cap N_{\tau_n}$ is empty, then a standard compactness argument that the inverse limit of finite nonempty sets is nonempty yields the existence of a chain $N_{\tau_1} \subset N_{\tau_2} \subset \ldots$ of such sets. And $\bigcup_{n=1}^\infty N_{\tau_n} = S_\sigma$ for some σ. Hence $P \cap S$ is empty. Therefore if P does not meet S, then P does not meet S_σ, and we have shown that $R[\{X_i\}]$ is Noetherian. It is clear that $R[\{X_i\}]_S$ is contained in $T = \bigcap R\langle X_{\sigma 1}, X_{\sigma 2}, \ldots \rangle$. To show that T is contained in $R[\{X_i\}]_S$, take $y \in T$, and let $A = \{f \in R[\{X_i\}] | fy \in R[\{X_i\}]\}$. Then A is an ideal of $R[\{X_i\}]$, and by the definition of T it follows that A meets each S_σ. By Lemma 10 and the compactness argument quoted above, A meets S, and hence $y \in R[\{X_i\}]_S$. This completes the proof of Theorem 11.

REMARK 12. If Λ is any totally ordered set and $\{X_\lambda | \lambda \in \Lambda\}$ is a set of indeterminates over a Noetherian ring R, then in analogy with Theorem 11, we can consider the group G of permutations of Λ. Each g in G induces a total order on Λ, and hence an ordering of the X_λ. Let S_g denote the multiplicative system of monic polynomials of $R[\{X_\lambda\}]$ with respect to the ordering defined by g. By Theorem 9, $R[\{X_\lambda\}]_{S_g}$ is Noetherian for each g in G. Indeed, in analogy with Theorem 11, if $S = \bigcap \{S_g | g \in G\}$, then $R[\{X_\lambda\}]_S = \bigcap_g R[\{X_\lambda\}]_{S_g}$, and this ring is again Noetherian. The proof for this result is similar to the proof given for Theorem 11, simply making use of the fact that $R[\{X_\lambda\}]$ is the direct union of the polynomial rings over R in a finite number of the X_λ, and the fact that any finite totally ordered set of cardinality n has the same order structure as the natural order on the set of positive integers $\leqslant n$.

We note that for R Noetherian, it follows from Theorem 5 that, in general, a quotient ring $R[\{X_\lambda\}]_S$ is Noetherian if $R[Y]_{S \cap R[Y]}$ is Noetherian for each countably infinite subset Y of $\{X_\lambda\}$.

References

1. J. W. Brewer and D. L. Costa, *Projective modules over some non-Noetherian polynomial rings*, J. Pure Appl. Algebra (to appear).

2. R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972. MR **55** #323.

3. R. Gilmer and W. Heinzer, *On the divisors of monic polynomials over a commutative ring*, Pacific J. Math. **78** (1978), 121–131.

4. I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR **40** #7234.

5. W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche*. VII. *Multiplikativ abgeschlossene Systeme von endlichen Idealen*, Math. Z. **48** (1943), 533–552. MR **5**, 33.

6. ______, *Jacobsonsche Ringe, Hilbertscher Nullstellensatz, Dimensionentheorie*, Math. Z. **54** (1951), 354–387. MR **13**, 903.

7. T. Y. Lam, *Serre's conjecture*, Lecture Notes in Math., vol. 635, Springer-Verlag, Berlin and New York, 1978.

8. R. Matsuda, *Infinite group rings*. III, Bull. Fac. Sci. Ibaraki Univ. Ser. A Math. **8** (1976), 1–45.

9. N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942), 286–295. MR **3**, 262.

10. M. Nagata, *Local rings*. Interscience Tracts on Pure and Appl. Math., no. 13, Interscience, New York, 1962. MR **27** #5790.

11. D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171. MR **55** #337.

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