THE NOETHERIAN PROPERTY FOR QUOTIENT RINGS OF INFINITE POLYNOMIAL RINGS

ROBERT GILMER\(^1\) AND WILLIAM HEINZER\(^2\)

ABSTRACT. Let \(X\) be an infinite set of indeterminates over the commutative Noetherian ring \(R\) with identity. We prove that the quotient ring of \(R[X]\) with respect to the multiplicative system of polynomials of unit content is also Noetherian. Moreover, we show that certain quotient rings of \(R[X]\) with respect to multiplicative systems of monic polynomials (where "monic" is appropriately defined) are also Noetherian.

Let \(X = \{X_\lambda\}_{\lambda \in \Lambda}\) be a set of indeterminates over \(R\), a commutative ring with identity. The content of a polynomial \(f \in R[X]\) is denoted by \(C(f)\) and is defined to be the ideal of \(R\) generated by the coefficients of \(f\); if \(C(f) = R\), then \(f\) has unit content. The quotient ring of \(R[X]\) with respect to the multiplicative set of polynomials of unit content is denoted by \(R(X)\). Ray Heitmann has asked\(^3\) if \(R(X)\) is Noetherian, provided the ring \(R\) is Noetherian. We show in Theorem 6 that the answer to this question is affirmative. We subsequently define other quotient rings of \(R[X]\), contained in \(R(X)\) and defined in terms of certain multiplicative sets of "monic" polynomials of \(R[X]\). We show the Noetherian property to be preserved also for these quotient rings of \(R[X]\).

The ring \(R(X)\) seems to have been first considered by Krull in [5], while Nagata introduced the notation \(R(X)\) in [10, p. 17]. The following result, which is a partial citation of Proposition 33.1 of [2], lists some basic properties of the ring \(R(X)\).

PROPOSITION 1. Let \(S\) be the set of elements of \(R[X]\) of unit content, and let \(\{M_\beta\}_{\beta \in \beta}\) be the set of maximal ideals of \(R\). For each \(\beta \in \beta\), denote by \(M_\beta[X]\) and \(M_\beta(X)\) the ideals of \(R[X]\) and \(R(X)\), respectively, generated by \(M_\beta\).

1. \(S = R[X] - (\bigcup_{\beta \in \beta} M_\beta[X])\).
2. \(\{M_\beta[X]\}\) is the family of ideals of \(R[X]\) maximal with respect to the property of failure to meet the multiplicative system \(S\); hence \(\{M_\beta(X)\}\) is the set of maximal ideals of \(R(X)\).
3. If \(Q\) is an ideal of \(R\), then \(QR(X) \cap R = Q\); if \(Q\) is \(P\)-primary in \(R\), then \(QR(X)\) is \(PR(X)\)-primary.

Presented to the Society November 12, 1977; received by the editors September 9, 1978.


Key words and phrases. Noetherian ring, polynomial of unit content, monic polynomial.

Research supported by NSF Grant MCS 75-06591.

Research supported by NSF Grant 7800798.

Private communication.

© 1979 American Mathematical Society

0002-9939/79/0000-0350/S02.75

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
For a prime ideal $P$ of a commutative ring, we use $h(P)$ to denote the height of $P$ ([2, p. 109], [10, p. 24]).

**Lemma 2.** If $R$ is Noetherian, if $T$ is a quotient ring of the polynomial ring $R[[X_\lambda]]$, and $P$ is a prime ideal of $T$, then $h(P) > h(P \cap R)$, and equality holds if and only if $P = (P \cap R)T$.

**Proof.** Since prime ideals of $T$ are the extensions of their contractions in $R[[X_\lambda]]$, it suffices to prove the result for $T = R[[X_\lambda]]$. Moreover, since prime ideals extend to prime ideals and contract to themselves with respect to polynomial ring extension, and since $R[[X_\lambda]]$ is the direct union of the polynomial rings over $R$ on finite subsets of $\{X_\lambda\}$, it suffices to prove the result for $T = R[X_1, \ldots, X_n]$, and hence just for the case of a polynomial ring in one variable over a Noetherian ring. The result in this case is well known ([6, §6], [4, Theorem 149]).

**Lemma 3.** If $R$ is Noetherian, if $T$ is a quotient ring of the polynomial ring $R[[X_\lambda]]$, and $P$ is a prime ideal of $T$ of finite height, then there exists a finite subset $\{X_1, \ldots, X_n\}$ of $\{X_\lambda\}$ such that $P = (P \cap R[X_1, \ldots, X_n])T$.

**Proof.** Again, since each prime ideal of $T$ is the extension of its contraction to $R[[X_\lambda]]$, we may assume that $T = R[[X_\lambda]]$. If $h(P) = r$, let $P_0 < P_1 < \cdots < P_r = P$ be a chain of prime ideals of $T$, and let $f_i \in P_i \setminus P_{i-1}$, $i = 1, \ldots, r$. The $f_i$ are polynomials in a finite subset $\{X_1, \ldots, X_n\}$ of $\{X_\lambda\}$ and $h(P) = h(P \cap R[X_1, \ldots, X_n])$. Hence, $P = (P \cap R[X_1, \ldots, X_n])T$ by Lemma 2.

**Theorem 4.** If $R$ is Noetherian, if $T$ is a quotient ring of the polynomial ring $R[[X_\lambda]]$, and $P$ is a prime ideal of $T$ of finite height, then $P$ is finitely generated.

**Proof.** By Lemma 3, $P = (P \cap R[X_1, \ldots, X_n])T$. Since $R$ is Noetherian, $R[X_1, \ldots, X_n]$ is Noetherian by the Hilbert Basis Theorem. Hence $P \cap R[X_1, \ldots, X_n]$, and therefore $P$, is finitely generated.

**Theorem 5.** If $R$ is Noetherian and $T$ is a quotient ring of the polynomial ring $R[[X_\lambda]]$, then $T$ is Noetherian if and only if each prime ideal of $T$ has finite height.

**Proof.** By Krull’s Principal Ideal Theorem, each prime ideal of a Noetherian ring has finite height. Therefore the condition is clearly necessary. If each prime ideal of $T$ has finite height, then, by Theorem 4, each prime ideal of $T$ is finitely generated, so by a theorem of Cohen ([4, p. 5] or [10, p. 8]) $T$ is Noetherian.

---

4If $R$ is not Noetherian, it can happen that $P = (P \cap R)[X]$ and $h(P) > h(P \cap R)$ [2, p. 364].
Theorem 6. If $R$ is Noetherian, then $R(X) = R(\{X_\lambda\}_{\lambda \in \Lambda})$ is also Noetherian.

Proof. By Theorem 5 and Proposition 1, it suffices to show for each maximal ideal $M$ of $R$ that $MR(\{X_\lambda\})$ has finite height. Since $R$ is Noetherian, $M$ is of finite height, and, by Lemma 2, $h(M) = h(MR(\{X_\lambda\}))$.

Assume that $Y$ is an indeterminate over the ring $R$. A ring closely related to $R(Y)$ is the quotient ring of $R[Y]$ with respect to the multiplicative system of monic polynomials over $R$. We denote this ring by $R\langle Y \rangle$; it arises in Quillen’s proof of the Serre Conjecture [11], and has been the object of some other recent investigation ([1], [7, Chapter IV], [8]). For a polynomial ring in two variables over $R$, say $R[X_1, X_2]$, a natural analogue of $R\langle Y \rangle$ is obtained by defining $R\langle X_1, X_2 \rangle$ to be the quotient ring of $R[X_1, X_2]$ at the multiplicative system of monic polynomials in $X_2$ over $R\langle X_1 \rangle$. By induction we define

$$R\langle X_1, \ldots, X_n \rangle = R\langle X_1, \ldots, X_{n-1} \rangle\langle X_n \rangle.$$  

Note that the definition of $R\langle X_1, \ldots, X_n \rangle$, unlike that of $R(X_1, \ldots, X_n)$, depends upon the order of the indeterminates. For example, it is proved in [3, Proposition 10] that $R\langle X_1, X_2 \rangle$ and $R\langle X_2, X_1 \rangle$ are equal as subrings of the total quotient ring of $R[X_1, X_2]$ if and only if $R$ is 0-dimensional.

In general, if $\Lambda$ is a totally ordered set and $\{X_\lambda\}_{\lambda \in \Lambda}$ is a set of indeterminates over $R$, then we define $R\langle \{X_\lambda\} \rangle$ to be the union of the directed set $\{R\langle X_\lambda, \ldots, X_\nu \rangle\}$ of subrings of the total quotient ring of $R(\{X_\lambda\})$, the union being taken over all finite subsets $\{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ of $\Lambda$. It is clear that $R\langle \{X_\lambda\} \rangle$ so defined is a regular quotient ring of $R(\{X_\lambda\})$. We proceed to show that $R\langle \{X_\lambda\} \rangle = R(\{X_\lambda\})_S$, where $S$ is the multiplicative system of “monic” polynomials in $R(\{X_\lambda\})$, defined in the following manner. The total order on $\Lambda$ induces, via the reverse lexicographic order, a total order on the set of monomials in the indeterminates $X_\lambda$, as follows. If $\lambda_1, \ldots, \lambda_n \in \Lambda$ are such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and if $e_1, \ldots, e_n, f_1, \ldots, f_n$ are nonnegative integers, then

$$X_{\lambda_1}^{e_1}X_{\lambda_2}^{e_2}\cdots X_{\lambda_n}^{e_n} < X_{\lambda_1}^{f_1}X_{\lambda_2}^{f_2}\cdots X_{\lambda_n}^{f_n}$$

if and only if $e_i < f_i$ for the largest integer $i$ such that the corresponding exponents are unequal. This order on monomials is compatible with multiplication. If $g$ is a nonzero element of $R(\{X_\lambda\})$, then $g$ can be written in the form

$$g = r_1Y_1 + \cdots + r_kY_k,$$
where \( r_i \in R, r_k \neq 0 \), and \( Y_1, \ldots, Y_k \) are monomials such that \( Y_1 < \cdots < Y_k \). We call \( Y_k \) the leading monomial of \( g \), \( r_k \) the leading coefficient of \( g \), and say that \( g \) is monic if \( r_k = 1 \). Let \( S \) be the set of monic polynomials in \( R[[X]] \). It is clear that \( S \) is a multiplicative system in \( R[[X]] \). Since \( S \cap R[X, \ldots, X_k] \) is the set of monic polynomials in \( R[X_1, \ldots, X_k] \), and since \( R\langle X \rangle \) is defined to be \( \bigcup \{ R\langle X_1, \ldots, X_k \rangle | \{X\}_k \text{ is a finite subset of } \Lambda \} \), it is sufficient to establish the equality \( R\langle X \rangle = R[[X]]_S \) in the case where \( \Lambda \) is finite. For the case of one variable, say \( X_1 \), \( R\langle X_1 \rangle = R[X_1]_S \) by definition. For \( n \) variables, say \( X_1, \ldots, X_n \), it is clear that \( R[X_1, \ldots, X_n]_S \) is contained in \( R\langle X_1, \ldots, X_n \rangle \), and proceeding by induction, we may assume that \( R\langle X_1, \ldots, X_{n-1} \rangle[X_n]_S \) is contained in \( R[X_1, \ldots, X_n]_S \). If \( f \) is a polynomial in \( R\langle X_1, \ldots, X_{n-1} \rangle[X_n] \) that is monic as a polynomial in \( X_n \) with coefficients in \( R\langle X_1, \ldots, X_{n-1} \rangle \), then we wish to show that \( f \) is a unit in \( R[X_1, \ldots, X_n]_S \). We have \( f = X_n^m + a_{m-1}X_n^{m-1} + \cdots + a_0 \) with the \( a_i \in R\langle X_1, \ldots, X_{n-1} \rangle \). By the induction hypothesis, there exists \( s \in S \cap R[X_1, \ldots, X_{n-1}] \) such that \( sa_i \in R[X_1, \ldots, X_{n-1}] \) for \( i = 1, \ldots, m - 1 \). If \( s \) has leading monomial \( Y \), then \( YX_n^m \) is the leading monomial of \( sf \in R[X_1, \ldots, X_n] \), and \( sf \) is a monic polynomial in \( R[X_1, \ldots, X_n] \). Therefore \( sf \in S \) and \( f \) is a unit of \( R[X_1, \ldots, X_n]_S \). We conclude that \( R\langle X \rangle = R[[X]]_S \).

We shall prove in Theorem 9 that the ring \( R\langle X \rangle \) is Noetherian if \( R \) is Noetherian; for finite dimensional \( R \), this follows readily from Theorem 5.

**Proposition 7.** Let \( \Lambda \) be a totally ordered set and let \( \{X_\lambda | \lambda \in \Lambda \} \) be a set of indeterminates over the finite dimensional Noetherian ring \( R \). Then \( R\langle \{X_\lambda | \lambda \in \Lambda \} \rangle \) is Noetherian.

**Proof.** By Theorem 5, it suffices to show that \( R\langle X \rangle \) is finite dimensional. If \( J \) is an \( n \)-dimensional Noetherian ring, it is known that \( J \langle Y \rangle \) again has dimension \( n \) ([1], [7, Proposition 1.2, Chapter IV]). Since \( R\langle X \rangle \) is the direct limit of the rings \( R\langle X_\lambda \rangle \) for finite subsets \( \{X_\lambda \}_{\lambda \in \Lambda} \) of \( \{X_\lambda \} \), and since the direct limit of rings of dimension \( n \) is a ring of dimension \( < n \), we conclude that \( R\langle X \rangle \) is Noetherian. (In fact, \( \dim R\langle X \rangle = n \).

**Remark 8.** If \( N \) is a multiplicative system in \( R \), then there exist canonical homomorphisms \( \varphi: R(X)_N \rightarrow R_N(X) \) and \( \psi: R\langle X \rangle_N \rightarrow R_N\langle X \rangle \). In general, these canonical homomorphisms are not surjective. This is the case even if \( N \) is the complement of a maximal ideal of \( R \). For example, if \( R \) is a polynomial ring in two variables over an algebraically closed field and \( N \) is the complement of a maximal ideal of \( R \), then it is easy to see that \( R(X)_N \) and \( R\langle X \rangle_N \) are properly contained in \( R_N(X) \) and \( R_N\langle X \rangle \) as subrings of the quotient field of \( R[X] \). Thus, to show \( R\langle X \rangle \) is Noetherian for \( R \) an infinite dimensional Noetherian ring, something other than just a naive localization argument and an application of Proposition 7 is necessary. We show below, however, that a modified localization argument does work.
Theorem 9. If $\Lambda$ is a totally ordered set and $\{X_\lambda|\lambda \in \Lambda\}$ is a set of indeterminates over the Noetherian ring $R$, then $R\langle\{X_\lambda\}\rangle$ is Noetherian.

Proof. Let $S$ be the multiplicative system of monic polynomials in $R\{\{X_\lambda\}\}$. We have $R\{\{X_\lambda\}\}_S = R\langle\{X_\lambda\}\rangle$, and to show that $R\langle\{X_\lambda\}\rangle$ is Noetherian, it suffices, by Theorem 5, to show that if $P$ is a prime ideal of $R\{\{X_\lambda\}\}$ not meeting $S$, then $P$ is of finite height. Consider the set $A$ consisting of zero and the set of leading coefficients of elements of $P$. It is clear that $A$ is closed under multiplication by elements of $R$. We note that $A$ is also closed under subtraction. For suppose $r_1$ and $r_2$ are the leading coefficients of $f_1$ and $f_2$ in $P$. Certainly $r_1 - r_2$ is in $A$ if $r_1 = 0$, $r_2 = 0$, or $r_1 = r_2$. Otherwise, if $m_1$ is the leading monomial of $f_1$, then $r_1 - r_2$ is the leading coefficient of $f_1m_2 - f_2m_1 \in P$. Therefore $A$ is an ideal of $R$, and since $P$ does not meet $S$, the ideal $A$ is proper. Let $M$ be a maximal ideal of $R$ containing $A$, and let $\varphi$ denote the canonical homomorphism of $R\{\{X_\lambda\}\}$ into $R_M\{\{X_\lambda\}\}$. We prove that the prime ideal $\varphi(P)R_M\{\{X_\lambda\}\}$ does not meet the multiplicative system of monic polynomials of $R_M\{\{X_\lambda\}\}$. Suppose not. Then there exist $f \in P$ and $r \in R \setminus M$ such that $\varphi(f)/\varphi(r)$ is a monic polynomial in $R_M\{\{X_\lambda\}\}$. If $Y$ is the leading monomial of $\varphi(f)/\varphi(r)$ in $R_M\{\{X_\lambda\}\}$, then there exists $t \in R \setminus M$ such that $tf$ is in $R\{\{X_\lambda\}\}$ and has leading monomial $Y$. If $u$ is the leading coefficient of $tf$, then $\varphi(u) = \varphi(rt)$. Since $rt \notin M$, we have $u \notin M$. But $tf \in P$ so that $u \in A$. This contradicts the fact that $A$ is contained in $M$. Therefore $\varphi(P)R_M\{\{X_\lambda\}\}$ extends to a proper ideal in $R_M\langle\{X_\lambda\}\rangle$. Since $R_M$ is a finite dimensional Noetherian ring, $R_M\langle\{X_\lambda\}\rangle$ is Noetherian by Proposition 7. Therefore $\varphi(P)R_M\{\{X_\lambda\}\}$ is a prime ideal of finite height. Since $h(P) = h(\varphi(P)R_M\{\{X_\lambda\}\})$, this completes the proof of Theorem 9.

If $\sigma$ is a nonidentity permutation on the set of positive integers, and $R$ is a ring of dimension greater than 0, then it follows from [3, Proposition 10] that $R\{X_1, X_2, \ldots\}$ and $R\{X_{\sigma_1}, X_{\sigma_2}, \ldots\}$ are distinct subrings of the total quotient ring of $R\{\{X_\lambda\}\}$, of course, these rings are $R$-isomorphic under the mapping taking $X_i$ to $X_{\sigma_i}$, and by Theorem 9, each is Noetherian if $R$ is Noetherian. It seems natural, therefore, to consider the ring $T = \cap \{R\langle X_{\sigma_1}, X_{\sigma_2}, \ldots\|\sigma$ is a permutation on the set of positive integers), and the multiplicative system $S$ of $R\{\{X_\lambda\}\}$ consisting of polynomials that are monic with respect to each ordering of the set of positive integers defined by a permutation $\sigma$. We proceed to show that $T = R\{\{X_\lambda\}\}_S$, and that this ring is Noetherian if $R$ is Noetherian.

Let $G_n$ denote the permutation group on $\{1, \ldots, n\}$, and for $\sigma \in G_n$, let $N_\sigma$ denote the multiplicative system of monic polynomials in $R\{X_1, \ldots, X_n\}$ under the ordering $\sigma 1 < \sigma 2 < \cdots < \sigma n$. Thus $R\{X_1, \ldots, X_n\}_N = R\langle X_{\sigma_1}, \ldots, X_{\sigma_n}\rangle$. Note that any monomial in $X_{\sigma_1}, \ldots, X_{\sigma_n}$ is in $N_\sigma$, and if $f$ is an element of $N_\sigma$ and $m$ is a monomial, then $mf$ is again in $N_\sigma$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Lemma 10. If $f_{\sigma}$ is an element of $N_{\sigma}$ for each $\sigma$ in $G_n$, then there exist monomials $m_{\sigma}$ in $X_1, \ldots, X_n$ such that $f = \sum m_{\sigma} f_{\sigma}$ is in each $N_{\sigma}$. In particular, if $A$ is an ideal of $R[X_1, \ldots, X_n]$ meeting each of the multiplicative systems $N_{\sigma}$, then $A$ contains an element $f$ such that $f$ is in each $N_{\sigma}$.

Proof. We proceed by induction on $n$, the case $n = 1$ being immediate. Let $\sigma_i \in G_n$ be such that $\sigma_i n = i$ for $i = 1, \ldots, n$. For a fixed $i$, consider all $t$ in $G_n$ such that $t i = i$. Using the induction hypothesis that the result is true for $n - 1$, we obtain a linear combination with monomial coefficients of the $f_t$ such that $t n = i$, say $g_t$, that is in every $N_t$ for which $t n = i$. It remains to show that there exist monomials, say $p_{\sigma_i}$ in $X_1, \ldots, X_n$ such that $\sum_{i=1}^n p_{\sigma_i} g_{\sigma_i}$ is in every $N_{\sigma_i}$. We choose $p_{\sigma_i} = X_{\sigma_i}^{e_{\sigma_i}}$, where $e_{\sigma_i}$ is a positive integer greater than the degree in $X_{\sigma_i}$ of any $g_{\sigma_i}$. It follows that $\sum_{i=1}^n X_{\sigma_i}^{e_{\sigma_i}} g_{\sigma_i}$ is in every $N_{\sigma_i}$.

Theorem 11. Let $R$ be a Noetherian ring and let $S$ be the multiplicative system in $R[[X_1]]$ of polynomials that are monic with respect to each ordering of the $X_i$. Then $R[[X_1]] S$ is again Noetherian, and is equal to $\bigcap \{ R<X_{\sigma_1},X_{\sigma_2},\ldots > | \sigma$ is a permutation on the set of positive integers $\}.$

Proof. For each permutation $\sigma$, let $S_{\sigma}$ denote the multiplicative system of polynomials of $R[[X_1]]$ that are monic with respect to the ordering $\sigma_1 < \sigma_2 < \ldots$. Then $S = \bigcap_{\sigma} S_{\sigma}$, and to show that $R[[X_1]] S$ is Noetherian, it suffices, by Theorem 5, to show that each prime ideal $P$ of $R[[X_1]]$ that does not meet $S$ is of finite height. If $P$ does not meet $S_{\sigma}$ for some $\sigma$, then it is clear that $P$ is of finite height, for $R[[X_1]] S_{\sigma} = R<X_{\sigma_1},X_{\sigma_2},\ldots >$ is Noetherian by Theorem 9. Hence $PR<X_{\sigma_1},X_{\sigma_2},\ldots >$, and therefore $P$, is of finite height. We show that if $P$ is a prime ideal of $R[[X_1]]$ that meets each $S_{\sigma}$, then $P$ meets $S$. Consider $R[X_1, \ldots, X_n]$ and the $n!$ multiplicative systems $N_{\tau}$ of $R[X_1, \ldots, X_n]$ associated with permutations $\tau$ of $\{1, \ldots, n\}$. For each $S_{\sigma}$ we have $S_{\sigma} \cap R[X_1, \ldots, X_n] = N_{\tau}$ for some $\tau$. If $P$ meets each of the $N_{\tau}$ on $R[X_1, \ldots, X_n]$, then, by Lemma 10, $P$ contains an element $f$ such that $f$ belongs to each $N_{\tau}$, and therefore to each $S_{\sigma}$, and hence to $S = \bigcap_{\sigma} S_{\sigma}$. But, if for each positive integer $n$, there is a multiplicative system $N_{\sigma_n}$ on $R[X_1, \ldots, X_n]$ such that $P \cap N_{\sigma_n}$ is empty, then a standard compactness argument that the inverse limit of finite nonempty sets is nonempty yields the existence of a chain $N_{\sigma_1} \subset N_{\sigma_2} \subset \ldots$ of such sets. And $\bigcup_{\sigma=1}^{\infty} N_{\sigma_n} = S_{\sigma}$ for some $\sigma$. Hence $P \cap S$ is empty. Therefore if $P$ does not meet $S$, then $P$ does not meet $S_{\sigma}$, and we have shown that $R[[X_1]]$ is Noetherian. It is clear that $R[[X_1]] S$ is contained in $T = \bigcap R<X_{\sigma_1},X_{\sigma_2},\ldots >$. To show that $T$ is contained in $R[[X_1]] S$, take $y \in T$, and let $A = \{ f \in R[[X_1]] | fy \in R[[X_1]] \}$. Then $A$ is an ideal of $R[[X_1]]$, and by the definition of $T$ it follows that $A$ meets each $S_{\sigma}$. By Lemma 10 and the compactness argument quoted above, $A$ meets $S$, and hence $y \in R[[X_1]] S$. This completes the proof of Theorem 11.

Remark 12. If $\Lambda$ is any totally ordered set and $\{X_\lambda | \lambda \in \Lambda\}$ is a set of indeterminates over a Noetherian ring $R$, then in analogy with Theorem 11,
we can consider the group $G$ of permutations of $\Lambda$. Each $g$ in $G$ induces a total order on $\Lambda$, and hence an ordering of the $X_\lambda$. Let $S_g$ denote the multiplicative system of monic polynomials of $R[(X_\lambda)]$ with respect to the ordering defined by $g$. By Theorem 9, $R[(X_\lambda)]_{S_g}$ is Noetherian for each $g$ in $G$. Indeed, in analogy with Theorem 11, if $S = \cap \{S_g | g \in G\}$, then $R[(X_\lambda)]_S = \cap_g R[(X_\lambda)]_{S_g}$, and this ring is again Noetherian. The proof for this result is similar to the proof given for Theorem 11, simply making use of the fact that $R[(X_\lambda)]$ is the direct union of the polynomial rings over $R$ in a finite number of the $X_\lambda$, and the fact that any finite totally ordered set of cardinality $n$ has the same order structure as the natural order on the set of positive integers $< n$.

We note that for $R$ Noetherian, it follows from Theorem 5 that, in general, a quotient ring $R[(X_\lambda)]_S$ is Noetherian if $R[Y]_{S \cap R[Y]}$ is Noetherian for each countably infinite subset $Y$ of $\{X_\lambda\}$.

References


Department of Mathematics, Florida State University, Tallahassee, Florida 32306
Department of Mathematics, Purdue University, West Lafayette, Indiana 47907