THE ESSENTIAL CLOSURE OF $C(X)$

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Abstract. Each archimedean $l$-group admits a unique essential closure, which is the $l$-group of continuous almost finite real-valued functions on some Stonean space; thus the $l$-group $C(X)$ of real-valued continuous functions on a topological space $X$ admits such an essential closure. In this note we will construct a natural embedding of $C(X)$ into its essential closure, making explicit the topological relationship between $X$ and the appropriate Stonean space.

1. Preliminaries. Throughout this paper $G$ will denote an archimedean lattice-ordered group ($l$-group). A general reference on $l$-groups is [2]. If $G$ is an $l$-subgroup of an $l$-group $H$, then $G$ is large in $H$ (or $H$ is an essential extension of $G$) if $C \cap G \neq 0$, for each convex $l$-subgroup $C$ of $H$. An archimedean $l$-group is essentially closed if it admits no proper archimedean essential extensions; $H$ is an essential closure of $G$ if $H$ is essentially closed and an essential extension of $G$. That each archimedean $l$-group admits a unique essential closure is due to Conrad [5]; this closure is of the form $D(Y)$, where $Y$ is a Stonean space (that is, compact Hausdorff and extremally disconnected). Here $D(Y) = \{ f: Y \to \mathbf{R}^*: f^{-1}(\mathbf{R}) \text{ is dense in } Y \text{ and } f \text{ is continuous} \}$, where $\mathbf{R}^* = \mathbf{R} \cup \{ \pm \infty \}$ is the two-point compactification of the real numbers $\mathbf{R}$. Bernau [1] first showed that any archimedean $l$-group could be embedded into such an $l$-group.

For $K \subseteq G$, let

$$K' = \{ g \in G: |g| \land |x| = 0, \text{ for } x \in K \}.$$

Then

$$P(G) = \{ C': C \text{ is a convex } l\text{-subgroup of } G \}$$

is the set of polars of $G$; $P(G)$ is a subset of the set of convex $l$-subgroups of $G$, and is a complete Boolean algebra, with set-theoretic intersection for meet, and ' for complementation. If $H$ is archimedean and $G \subseteq H$, then $G$ is large in $H$ precisely when the natural intersection map $C \to C \cap G$ is a Boolean algebra isomorphism between $P(H)$ and $P(G)$ [4].

Given a Tychonoff space $X$, let $\Theta X$ be the set of all regularly open ultrafilters on $X$. If $U$ is a regularly open subset of $X$, let $\Theta (U) = \{ p \in \Theta X: U \in p \}$. The set of all such $\Theta (U)$ forms a base for the Stone topology on $\Theta X$, which makes $\Theta X$ a Stonean space [6]. Furthermore, $\Theta : \mathcal{R}(X) \to \mathcal{R}(\Theta X)$
is a Boolean algebra isomorphism between the regularly open subsets of $X$ and of $\Theta X$ [3, p. 40]. Now let $\omega X = \{ p \in \Theta X : p$ is fixed$\}$, which is called the absolute of $X$. Then $\omega X$ is a dense subset of $\Theta X$ [6], and so $\beta(\omega X) = \Theta X$. Define $\pi : \omega X \to X$ by letting $\pi(p)$ be the (unique) point to which $p$ converges. Then $\pi$ is a continuous function [6].

2. The embedding. We will need the following result which characterizes polars topologically:

**Proposition.** Let $X$ be a Tychonoff space. Let $G = C(X)$ (or $D(X)$, where $X$ is Stonean). Then the map $\tau(X) : P(G) \to D(X)$ defined by

\[ \tau(X)(C) = \text{Interior}(\text{Closure}\{ x \in X : f(x) \neq 0, \text{some } f \in C \}) \]

is a Boolean algebra isomorphism.

This proposition is well known and its proof will be omitted.

**Theorem.** If $X$ is a Tychonoff space, then $\alpha : C(X) \to D(\Theta X)$, defined so that the following diagram commutes for all $f \in C(X)$, is a large $l$-embedding of $C(X)$ into its essential closure:

\[ \beta(\omega X) = \Theta X \]

\[ \omega X \xrightarrow{\pi} X \xrightarrow{f} R \xrightarrow{\iota} R^* \]

\[ \alpha(f) = \beta(\iota \circ f \circ \pi) \]

**Proof.** Since $\alpha(f)^{-1}(R)$ is an open subset of $\Theta X$ which contains $\omega X$ and is consequently dense, $\alpha(f)$ is an element of $D(\Theta X)$. Given $f, g \in C(X)$, $\alpha(f) + \alpha(g)$ is defined as the unique extension of $\alpha(f)|U + \alpha(g)|U$ to $\Theta X$, where $U = \alpha(f)^{-1}(R) \cap \alpha(g)^{-1}(R)$, an open dense set. (This unique extension exists because $X$ is Stonean.) But $\alpha(f) + \alpha(g)$ then agrees with $\alpha(f + g)$ on $\omega X$, a dense subset of $\Theta X$, and so $\alpha(f) + \alpha(g) = \alpha(f + g)$. A similar argument shows that $\alpha$ preserves the lattice operations. Since $\alpha$ is clearly monic, it remains to show that $C(X)$ (identified now with $\alpha(C(X))$) is large in $D(\Theta X)$. But

\[ \tau(X)^{-1} \Theta^{-1} \tau(\Theta X) : P(D(\Theta X)) \to P(C(X)) \]

is a Boolean algebra isomorphism, and

\[ \tau(X)^{-1} \Theta^{-1} \tau(\Theta X)(C) = C \cap C(X). \]

**References**


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