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MODULAR GROUP ALGEBRAS OF TOTALLY PROJECTIVE p-PRIMARY GROUPS

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ABSTRACT. Let F be a field of characteristic $p > 0$ and let G be a totally projective abelian p-group of countable p-length. If $FG \cong FH$ for some group H, then it is shown that $G \cong H$.

Let F be a field of characteristic $p > 0$ and let G be an (infinite) abelian p-group. The question of whether $FG \cong FH$ implies that $G \cong H$ has been considered by Berman and Mollov [**1**] and May [**5**]. (For the finite non-abelian case, see Passman [**8**].) It is shown in [**1**], [**5**] and by Dubois and Sehgal [**2**] that the Ulm-Kaplansky invariants of G and H must be equal. Thus one can easily settle the case when G is countable. Since the totally projective p-groups form the largest natural class of abelian p-groups that are determined by their Ulm-Kaplansky invariants, one would like to know in general whether H must be totally projective if G is. In this direction, Berman and Mollov give an affirmative answer if G is a direct sum of cylic groups. By utilizing a theorem of Hill [**4**], we shall extend these results to p-groups of countable p-length. Precisely, we shall prove:

THEOREM 1. *Let R be a commutative ring with identity in which p is not invertible, and let Ω be the first uncountable ordinal. Let G be an abelian p-group of p-length $< \Omega$ whose reduced part is totally projective. If H is a group such that RG and RH are isomorphic algebras, then H is isomorphic to G.*

To deduce Theorem 1, we shall prove several facts about the group of units of RG. For any commutative ring R with identity and group G, let $U(RG)$ denote the group of units of RG that have augmentation 1 (i.e., the coefficients sum to 1). We prove:

THEOREM 2. *Let R be a commutative ring with identity of characteristic p that is perfect (i.e., every element is a pth power). Let G be an abelian p-group of p-length $\leqslant \Omega$ whose reduced part is totally projective. Then $U(RG)$ is an abelian p-group of the same p-length as G, and whose reduced part is totally projective. Moreover, G is a direct factor of $U(RG)$.*

We remark that Mollov [**6**], [**7**] has investigated the structure of unit groups under certain restrictions.

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Before proving two preliminary lemmas, we must briefly discuss p-heights. Groups will be written multiplicatively, thus we shall use G^{p^σ} (σ an ordinal) to denote the subgroup of G consisting of elements with p-heights $\geqslant \sigma$. Suppose that R is a commutative ring with identity of characteristic p, and let G be an abelian p-group. Then $U(RG)$ is a p-group and consists precisely of the elements of RG of augmentation 1. Suppose further now that R is perfect. Then $U(RG)^p = U(R(G^p))$, thus it follows that $U(RG)^{p^\sigma} = U(R(G^{p^\sigma}))$ for every ordinal σ. Let $\alpha \in U(RG)$ and consider the support of α, i.e. those $g \in G$ that appear nontrivially in α. Clearly the p-height of α in $U(RG)$ is the minimum of the p-heights in G of the elements in the support of α. In particular, if we regard G as a subgroup of $U(RG)$, then it is an isotype subgroup. We also observe that the p-length of $U(RG)$ is clearly equal to the p-length of G.

LEMMA 1. *Let R be a perfect commutative ring with identity of characteristic p, and let G be an abelian p-group.*

(a) *Let A be a finite subgroup of G, and let N be a subgroup of $U(RA)$. Then N is a nice subgroup of $U(RG)$.*

(b) *If G is countable, then the reduced part of $U(RG)$ is totally projective.*

PROOF. Let $\alpha \in U(RG)$ and let $S = \{ga \mid g \in (\text{support } \alpha),\ a \in A\}$. If $\beta \in N$, then the p-height of $\alpha\beta$ equals the p-height of some element in the finite set S. Therefore β can be chosen to maximize this p-height. Hence N is nice in $U(RG)$.

Now suppose that G is countable, and let $A_1 \subseteq A_2 \subseteq \cdots$ be finite subgroups of G with $\bigcup_{i<\omega} A_i = G$. For each i, we can choose a family $\{N_{i\alpha}\}$ of subgroups of $U(RG)$ indexed by an initial segment of ordinals such that $U(RA_i) \subseteq N_{i\beta} \subseteq N_{i\alpha} \subseteq U(RA_{i+1})$ if $\beta < \alpha$, $N_{i\alpha} = \bigcup_{\beta<\alpha} N_{i\beta}$ if α is a limit ordinal,

$$N_{i0} = U(RA_i), \qquad \bigcup_{\alpha} N_{i\alpha} = U(RA_{i+1}), \quad \text{and} \quad (N_{i,\alpha+1} : N_{i\alpha}) = p.$$

Each $N_{i\alpha}$ is nice in $U(RG)$ by (a), therefore $\bigcup_{i<\omega}\{N_{i\alpha}\}$ is a nice decomposition series for $U(RG)$. It follows that the reduced part of $U(RG)$ is totally projective.

LEMMA 2. *Let R be a commutative ring with identity and let G be an abelian group. Suppose that there exists a group B such that G is isomorphic to a direct factor of $U(RB)$. Then G is a direct factor of $U(RG)$.*

PROOF. Let G be isomorphic to the direct factor V of $U(RB)$. Regarding V as a subgroup of $U(RB)$, the isomorphism $G \to V$ induces a ring homomorphism $RG \to RB$, and hence a homomorphism $U(RG) \to U(RB)$. By composing with a projection of $U(RB)$ onto V, we obtain a homomorphism $U(RG) \to V$ that maps G isomorphically to V. Thus G is a direct factor of $U(RG)$.

PROOF OF THEOREM 2. We have already observed that $U(RG)$ is an abelian

p-group of the same p-length as G. We shall now show that the reduced part of $U(RG)$ is totally projective by using induction on $|G|$. The countable case is done by Lemma 1, therefore we may suppose that G is uncountable. The assumption that G has totally projective reduced part of p-length $\leqslant \Omega$ means precisely that G is isomorphic to a direct sum of countable p-groups (see [**3**, Theorem 82.4]). Let τ be the first ordinal with $|\tau| = |G|$. We may assume that $G = \amalg_{\beta<\tau} A_\beta$, where each A_β is countable. For each $\alpha < \tau$, put $G_\alpha = \amalg_{\beta<\alpha} A_\beta$ and $U_\alpha = U(RG_\alpha)$. (Note that $G_0 = 1$.) The projection $G_{\alpha+1} \to G_\alpha$ with kernel A_α induces a surjective map $U_{\alpha+1} \to U_\alpha$ that is idempotent. Thus we obtain an inner direct splitting $U_{\alpha+1} = U_\alpha \times K_\alpha$. It is easily seen that $U(RG) = \amalg_{\alpha<\tau} K_\alpha$. But $|\alpha + 1| < |G|$, thus $|U_{\alpha+1}| < |G|$, and consequently the reduced part of $U_{\alpha+1}$ is totally projective by induction. It now follows that $U(RG)$ has totally projective reduced part.

To show that G is a direct factor of $U(RG)$ let $B = \amalg_{i<\omega} G$. Then the reduced part of $U(RB)$ is totally projective by what we have just shown. Let $f_\sigma(G)$ denote the σth Ulm-Kaplansky invariant of G. We know that $f_\sigma(B) \leqslant f_\sigma(U(RB))$ since B is isotype in $U(RB)$, therefore $f_\sigma(U(RB)) = f_\sigma(G) + f_\sigma(U(RB))$ for every ordinal σ. Moreover, if D_G and D_U are the maximal divisible subgroups of G and $U(RB)$, respectively, then $D_U \cong D_G \times D_U$. Hence, $U(RB) \cong G \times U(RB)$ since we are dealing with totally projective reduced parts. By Lemma 2, we conclude that G is a direct factor of $U(RG)$.

Since the complement of G in $U(RG)$ has totally projective reduced part, one could determine it up to isomorphism by computing the rank of its maximal divisible subgroup and its Ulm-Kaplansky invariants in terms of R and the Ulm-Kaplansky invariants of G. We remark that this is a feasible computation, but we shall not discuss it here.

Proof of Theorem 1. Our assumptions on R guarantee that there is a homomorphism (not necessarily surjective) $R \to F$, where F is an algebraically closed field of characteristic p. It follows that $FG \cong FH$. One easily sees that H must be an abelian p-group since a torsion-free element in H would be transcendental over F, while an element of order relatively prime to p would yield a nontrivial idempotent in FH. Moreover, the maximal divisible subgroups of G and H are isomorphic by reference to [**5**, Corollary 7].

We may assume that the isomorphism $FG \cong FH$ preserves augmentations, hence $U(FG) \cong U(FH)$. Theorem 2 allows us to conclude that the reduced part of $U(FH)$ is totally projective of countable p-length. Thus the reduced part of H is isomorphic to an isotype subgroup of countable p-length in the reduced part of $U(FH)$. It now follows immediately from a result of Hill [**4**, Theorem 1] that the reduced part of H is totally projective. As was mentioned in the introduction, the Ulm-Kaplansky invariants of G and H are known to be equal; therefore $G \cong H$.

We conclude with several questions and remarks. Does Theorem 1 hold if G has p-length $\geqslant \Omega$? In this case, we no longer have Hill's theorem to apply.

Even more, there is uncertainty whether Theorem 2 holds if the p-length is $> \Omega$ since G would not be a direct sum of countable groups. If one tries to prove Theorem 2 by induction on the p-length of G, the difficult step seems to occur at isolated ordinals. As a final question, if G is a torsion-complete abelian p-group, then is H torsion-complete? If so, then one could conclude G and H would be isomorphic since the Ulm-Kaplansky invariants serve to classify torsion-complete groups.

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