SEPARATED $G_a$-ACTIONS

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Abstract. Let $X$ be an open subvariety of an affine variety, i.e. a quasi-affine variety, over an algebraically closed field, and suppose the additive algebraic group $G_a$ acts on $X$. Then a geometric quotient of $X$ by $G_a$ exists if and only if every point $x$ of $X$ has a $G_a$-stable open neighborhood $U$ such that the morphism $G_a \times X \rightarrow U \times U$ which sends $(t, u)$ to $(tu, u)$ has closed image and finite fibres.

Let $X$ be a normal quasi-affine algebraic variety over the algebraically closed field $k$ on which the unipotent algebraic group $G$ acts nontrivially. This paper gives a sufficient geometric condition for the action of $G$ on $X$, for the existence of a geometric quotient $X/G$ of $X$ by $G$ in the sense of [1, p. 724]. To describe the condition, we need the following terminology: let $\Psi_X: G \times X \rightarrow X \times X$ be the morphism which sends $(t, x)$ to $(tx, x)$, and let $D_x$ denote the image of $\Psi_X$. (If $Y$ is a geometric quotient of $X$ by $G$ and $p: X \rightarrow Y$ is the quotient map, then $D_x = p^{-1}(\Delta_y)$, where $\Delta_Y \subseteq Y \times Y$ is the diagonal.) We say that the action of $G$ on $X$ is quasi-finite if $\Psi_X$ is a quasi-finite morphism, proper if $\Psi_X$ is a proper morphism, separated if $D_x$ is closed, and locally separated if every point $x$ of $X$ has a $G$-stable open neighborhood $U$ such that $D_U$ is closed in $U \times U$. Then our result is that if the action of $G$ on $X$ is quasi-finite and locally separated, $X/G$ exists. In particular, if $G = G_a$, then $X/G_a$ exists if and only if the action is quasi-finite and locally separated.

To place this result in perspective, we recall that for $X/G_a$ to exist it is necessary that the action be quasi-finite [1, p. 723] but this is not sufficient [1, Example 1, p. 727]. If the action is proper, $X/G_a$ exists [1, Theorem 4, p. 725] but this sufficient condition is not necessary [1, Example 2, p. 727].

The proof of the result is based on the idea of a "Sheshadri cover", [1, Theorem 1, p. 724]: if the action of $G$ on $X$ is quasi-finite, then there is a normal quasi-affine variety $Z$ on which $G$ acts locally trivially and on which a finite group $\Gamma$ operates such that the $G$ and $\Gamma$ actions on $Z$ commute, and a finite $G_a$-equivariant morphism $Z \rightarrow X$ displaying $X$ as the quotient of $Z$ by $\Gamma$. Since the action of $G$ on $Z$ is locally trivial, a geometric quotient $W$ of $Z$ by $G$ exists, and $\Gamma$ acts on $W$. If $X/G$ exists, it must equal $W/\Gamma$. We show that if the action of $G$ on $X$ is locally separated, then $W/\Gamma$, and hence $X/G$, exist. In general, $W$ is only a prevariety and not a variety, so we begin with a

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result on quotients of prevarieties by finite groups. We retain throughout the notational conventions of this introduction; in particular, all varieties and prevarieties are over \( k \).

**Definition.** A morphism \( f : W \to Y \) of prevarieties is a *pre-immersion* if every \( w \) in \( W \) has an affine open neighborhood \( U \) such that \( f(U) \) is open in \( Y \) and \( f : U \to f(U) \) is an isomorphism.

**Lemma.** Let \( W \) be a prevariety and \( V \) a normal affine variety, and let \( \Gamma \) be a finite group acting on \( W \) and \( V \). Let \( f : W \to V \) be a \( \Gamma \)-equivariant pre-immersion such that, for each \( w \) in \( W \), \( f^{-1}(w) \subseteq \Gamma \cdot w \). Then the geometric quotient \( W/\Gamma \) exists.

**Proof.** We need to show that each \( w \) in \( W \) has a \( \Gamma \)-stable open neighborhood \( U \) such that \( U/\Gamma \) exists. Choose an affine open neighborhood \( U' \) of \( w \) such that \( f\mid U' \) is an isomorphism. Let \( U = \bigcup \{ \gamma \cdot U' \mid \gamma \in \Gamma \} \). Since \( V \) is affine, \( V/\Gamma \) exists; let \( p : V \to V/\Gamma \) be the quotient morphism. We claim that \( (pf)(U) \) is a geometric quotient of \( U \) by \( \Gamma \). First, \( (pf)(U) = pf(U') \) is open in \( V/\Gamma \) and \( pf : U \to pf(U) \) is constant on \( \Gamma \)-orbits. Next, let \( x, y \in U \) and suppose \( pf(x) = pf(y) \). Then since \( V/\Gamma \) is a geometric quotient of \( V \) by \( \Gamma \), there is a \( \gamma \in \Gamma \) such that \( f(x) = \gamma f(y) = f(\gamma y) \). Thus \( x \in f^{-1}(\gamma y) \) so there is a \( \tau \in \Gamma \) such that \( x = \tau \gamma y \). Thus \( pf \) is an orbit map, and hence a geometric quotient morphism.

We can now establish the main result.

**Theorem.** Let \( X \) be a normal quasi-affine variety on which the unipotent algebraic group \( G \) acts nontrivially. If the action is quasi-finite and locally separated, a geometric quotient of \( X \) by \( G \) exists.

**Proof.** Suppose that the action of \( G \) on \( X \) is quasi-finite and locally separated. Let \( p : Z \to X \) be a Seshadri cover as above and let \( W = Z/G \). \( \Gamma \) denotes the finite group acting on \( Z \) and \( W \) with \( Z/\Gamma = X \). Let \( \{ W_i \mid 1 \leq i \leq n \} \) be an affine open cover of \( W \). Exactly as in the proof of [1, Proposition 3, p. 724], we can find a normal affine variety \( V \) and a pre-immersion \( f : W \to V \) such that \( f\mid W_i \) is an isomorphism onto the open subset \( f(W_i) \) of \( V \) for each \( i \), and such that \( \Gamma(W_i, O_{W_i}) = f^*(\Gamma(V, O_V)[1/b_i]) \) for appropriate \( b_i \). Moreover, the construction can be made so that \( \Gamma \) acts on \( V \) and \( f \) is \( \Gamma \)-equivariant. We note that \( f^*(\Gamma(V, O_V)) \) is contained in \( \Gamma(Z, O_Z)^G \). Let \( D_Z = \{(x, y) \in Z \times Z \mid h(x) = h(y) \text{ for all } h \in f^*(\Gamma(V, O_V))\} \). We claim that \( D_Z \) is the closure of \( D_Z \). Before establishing the claim, we show how the claim implies the existence of \( X/G \). Since the existence of the quotient is local on \( X \) in the obvious sense, we may assume the action of \( G \) on \( X \) is separated. Then \( D_X \) is closed in \( X \times X \), so \( p^{-1}(D_X) \) is closed in \( Z \times Z \), and thus contains \( D_Z \). Since \( p(D_Z) = D_X \), it follows that

\[
p^{-1}(D_X) = \{ (\gamma tz, z) \mid z \in Z, \gamma \in \Gamma, t \in G \}.
\]
Let $q: Z \to W$ be the quotient map, and let $w_1, w_2 \in W$ such that $f(w_1) = f(w_2)$. Let $z_i \in Z$ such that $q(z_i) = w_i$. Then $(z_1, z_2) \in D_Z$, so there is $t \in G$ and $\gamma \in \Gamma$ with $z_1 = \gamma t z_2$. Then

$$w_1 = q(z_1) = q(\gamma t z_2) = \gamma q(t z_2) = \gamma w_2.$$  

Thus the hypotheses of the lemma are satisfied, and $W/\Gamma$, hence $X/G$, exist.

It remains to establish the above claim. To see that $D_Z$ is closed, we observe that $\overline{D_Z}$ is the intersection of the zeros of the functions on $Z \times Z$ given by $(z_1, z_2) \mapsto h(z_1) - h(z_2)$ as $h$ varies over $f^*(\Gamma(V, O_V))$, and we also note that $D_Z$ is contained in $\overline{D_Z}$. Moreover, $D_Z$ and $\overline{D_Z}$ are $G \times G$-stable subsets of $Z \times Z$, so to show that $D_Z$ is dense in $\overline{D_Z}$ we can establish the corresponding fact about their images in $W \times W$. The image of $D_Z$ is $\Delta_w$, and the image of $\overline{D_Z}$ is $f^{-1}(\Delta_v)$. To show that $\Delta_w$ is dense in $f^{-1}(\Delta_v)$, we show that for each pair $(i, j)$, the closure in $W_i \times W_j$ of $\Delta_w \cap (W_i \times W_j)$ is $f^{-1}(\Delta_v) \cap (W_i \times W_j)$. Let $A_i = \Gamma(W_i, O_w)$, let $A_j = \Gamma(W_j, O_w)$ and let $A_{ij} = \Gamma(W_i \cap W_j, O_w)$. The inclusion $\Delta_w \cap (W_i \times W_j)$ induces the ring map $A_i \otimes_k A_j \to A_{ij}$ which sends $a \otimes b$ to $(a|W_i \cap W_j)(b|W_i \cap W_j)$. Let $I$ be the ideal of this ring map. The common zeros of $I$ in $W_i \times W_j$ is the closure of $\Delta_w \cap (W_i \times W_j)$. Suppose $\sum a_i \otimes e_i \in I$, $a_i \in A_i$, $e_i \in A_i$. We can choose a common denominator $d$ for the $e_i$, so $e_i = c_i/d$ with $c_i \in f^*(\Gamma(V, O_V))$ and $d = (\sum b_i)^d$, where $b_i$ is as above. Since $\sum a_i c_i = 0$,

$$\sum a_i \otimes e_i = \sum a_i \otimes c_i - \left(\sum a_i c_i \otimes \frac{1}{d}\right) = \sum (a_i \otimes 1)(1 \otimes e_i - c_i \otimes \frac{1}{d}).$$

If $(x, y)$ is in $f^{-1}(\Delta_v) \cap (W_i \times W_j)$, then $d(y) \neq 0$ and $c_i(x)/d(y)$ and

$$1 \otimes e_i - c_i \otimes \frac{1}{d})(x, y) = 0.$$  

It follows that $f^{-1}(\Delta_v) \cap (W_i \times W_j)$ is contained in the common zeros of $I$, namely the closure of $\Delta_w \cap (W_i \times W_j)$. The reverse inclusion follows from the fact that $f^{-1}(\Delta_v)$ is closed. This completes the proof of the theorem.

**Corollary.** Let $X$ be a normal quasi-affine variety with nontrivial $G_a$ action. Then $X/G_a$ exists if and only if the action is quasi-finite and locally separated.

**Proof.** If $X/G_a$ exists, the action is necessarily quasi-finite, as noted above, and the inverse images in $X$ of an affine open cover of $X/G_a$ provides a cover of $X$ by $G_a$-stable open sets on which the action is separated. The other implication is the theorem.

We close with the following remarks.

**Remark 1.** The above theorem implies the result, mentioned above, that a proper $G_a$-action on a quasi-affine variety $X$ has a quotient [1, Theorem 4, p. 725]: since $\Psi_X$ is proper, its image $D_X$ is closed, so the action is separated, and since $\Psi_X$ is proper between quasi-affines, it has finite fibres, so the action is quasi-finite. Thus a quotient exists.
Remark 2. Example 1 of [1, p. 727] provides an example of a quasi-finite $G_a$ action for which no quotient exists, hence, by the theorem, is an example of a quasi-finite nonlocally separated action. Example 2 of [2] proves an example of a nonproper $G_a$-action on a nonsingular affine with an affine quotient, and hence is an example of a quasi-finite separated action that is not proper.

Remark 3. The result has an analogue for the multiplicative group $G_m$: let $X$ be a quasi-affine variety with $G_m$ action. We can choose a $G_m$-equivariant open immersion of $X$ into an affine variety $V$ with $G_m$-action. The complement of $X$ in $V$ is given by the zeros of a $G_m$-stable ideal $I$ of the coordinate ring of $V$, and hence $I$ is generated by semi-invariants. The complements of the zeros of these semi-invariants gives an open cover of $X$ by $G_m$-stable open affines. For $x$ in $X$, the orbit of $x$ is $(X \times \{x\}) \cap D_x$. Thus if the action of $G_m$ on $X$ is separated, $D_x$ is closed so the orbits on $X$ are closed and hence a geometric quotient $X/G_m$ exists by [3, 1.3, p. 30].

References

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