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SEPARATED G_a-ACTIONS

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ABSTRACT. Let X be an open subvariety of an affine variety, i.e. a quasi-affine variety, over an algebraically closed field, and suppose the additive algebraic group G_a acts on X. Then a geometric quotient of X by G_a exists if and only if every point x of X has a G_a-stable open neighborhood U such that the morphism $G_a \times U \to U \times U$ which sends (t, u) to (tu, u) has closed image and finite fibres.

Let X be a normal quasi-affine algebraic variety over the algebraically closed field k on which the unipotent algebraic group G acts nontrivially. This paper gives a sufficient geometric condition for the action of G on X, for the existence of a geometric quotient X/G of X by G in the sense of [**1**, p. 724]. To describe the condition, we need the following terminology: let Ψ_X: $G \times X \to X \times X$ be the morphism which sends (t, x) to (tx, x), and let D_X denote the image of Ψ_X. (If Y is a geometric quotient of X by G and p: $X \to Y$ is the quotient map, then $D_X = p^{-1}(\Delta_Y)$, where $\Delta_Y \subseteq Y \times Y$ is the diagonal.) We say that the action of G on X is *quasi-finite* if Ψ_X is a quasi-finite morphism, *proper* if Ψ_X is a proper morphism, *separated* if D_X is closed, and *locally separated* if every point x of X has a G-stable open neighborhood U such that D_U is closed in $U \times U$. Then our result is that if the action of G on X is quasi-finite and locally separated, X/G exists. In particular, if $G = G_a$, then X/G_a exists if and only if the action is quasi-finite and locally separated.

To place this result in perspective, we recall that for X/G_a to exist it is necessary that the action be quasi-finite [**1**, p. 723] but this is not sufficient [**1**, Example 1, p. 727]. If the action is proper, X/G_a exists [**1**, Theorem 4, p. 725] but this sufficient condition is not necessary [**1**, Example 2, p. 727].

The proof of the result is based on the idea of a "Sheshadri cover", [**1**, Theorem 1, p. 724]: if the action of G on X is quasi-finite, then there is a normal quasi-affine variety Z on which G acts locally trivially and on which a finite group Γ operates such that the G and Γ actions on Z commute, and a finite G_a-equivariant morphism $Z \to X$ displaying X as the quotient of Z by Γ. Since the action of G on Z is locally trivial, a geometric quotient W of Z by G exists, and Γ acts on W. If X/G exists, it must equal W/Γ. We show that if the action of G on X is locally separated, then W/Γ, and hence X/G, exist. In general, W is only a prevariety and not a variety, so we begin with a

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result on quotients of prevarieties by finite groups. We retain throughout the notational conventions of this introduction; in particular, all varieties and prevarieties are over k.

DEFINITION. A morphism $f\colon W \to Y$ of prevarieties is a *pre-immersion* if every w in W has an affine open neighborhood U such that $f(U)$ is open in Y and $f\colon U \to f(U)$ is an isomorphsim.

LEMMA. *Let W be a prevariety and V a normal affine variety, and let Γ be a finite group acting on W and V. Let $f\colon W \to V$ be a Γ-equivariant pre-immersion such that, for each w in W, $f^{-1}f(w) \subseteq \Gamma \cdot w$. Then the geometric quotient W/Γ exists.*

PROOF. We need to show that each w in W has a Γ-stable open neighborhood U such that U/Γ exists. Choose an affine open neighborhood U' of w such that $f|U'$ is an isomorphism. Let $U = \bigcup \{\gamma \cdot U' | \gamma \in \Gamma\}$. Since V is affine, V/Γ exists; let $p\colon V \to V/\Gamma$ be the quotient morphism. We claim that $(pf)(U)$ is a geometric quotient of U by Γ. First, $(pf)(U) = pf(U')$ is open in V/Γ and $pf\colon U \to pf(U)$ is constant on Γ-orbits. Next, let $x,y \in U$ and suppose $pf(x) = pf(y)$. Then since V/Γ is a geometric quotient of V by Γ, there is a $\gamma \in \Gamma$ such that $f(x) = \gamma f(y) = f(\gamma y)$. Thus $x \in f^{-1}f(\gamma y)$ so there is a $\tau \in \Gamma$ such that $x = \tau\gamma y$. Thus pf is an orbit map, and hence a geometric quotient morphism.

We can now establish the main result.

THEOREM. *Let X be a normal quasi-affine variety on which the unipotent algebraic group G acts nontrivially. If the action is quasi-finite and locally separated, a geometric quotient of X by G exists.*

PROOF. Suppose that the action of G on X is quasi-finite and locally separated. Let $p\colon Z \to X$ be a Seshadri cover as above and let $W = Z/G$. Γ denotes the finite group acting on Z and W with $Z/\Gamma = X$. Let $\{W_i | 1 \leqslant i \leqslant n\}$ be an affine open cover of W. Exactly as in the proof of [**1**, Proposition 3, p. 724], we can find a normal affine variety V and a pre-immersion $f\colon W \to V$ such that $f|W_i$ is an isomorphism onto the open subset $f(W_i)$ of V for each i, and such that $\Gamma(W_i, O_W) = f^*(\Gamma(V, O_V)[1/b_i])$ for appropriate b_i. Moreover, the construction can be made so that Γ acts on V and f is Γ-equivariant. We note that $f^*(\Gamma(V, O_V))$ is contained in $\Gamma(Z, O_Z)^G$. Let $\bar{D}_Z = \{(x,y) \in Z \times Z | h(x) = h(y)$ for all $h \in f^*(\Gamma(V, O_V))\}$. We claim that $\bar{D}_Z$ is the closure of D_Z. Before establishing the claim, we show how the claim implies the existence of X/G. Since the existence of the quotient is local on X in the obvious sense, we may assume the action of G on X is separated. Then D_X is closed in $X \times X$, so $p^{-1}(D_X)$ is closed in $Z \times Z$, and thus contains $\bar{D}_Z$. Since $p(D_Z) = D_X$, it follows that

$$p^{-1}(D_X) = \{(\gamma t z, z) | z \in Z, \gamma \in \Gamma, t \in G\}.$$

Let $q\colon Z \to W$ be the quotient map, and let $w_1, w_2 \in W$ such that $f(w_1) = f(w_2)$. Let $z_i \in Z$ such that $q(z_i) = w_i$. Then $(z_1, z_2) \in \overline{D}_Z$, so there is $t \in G$ and $\gamma \in \Gamma$ with $z_1 = \gamma t z_2$. Then

$$w_1 = q(z_1) = q(\gamma t z_2) = \gamma q(t z_2) = \gamma w_2.$$

Thus the hypotheses of the lemma are satisfied, and W/Γ, hence X/G, exist.

It remains to establish the above claim. To see that $\overline{D}_Z$ is closed, we observe that $\overline{D}_Z$ is the intersection of the zeros of the functions on $Z \times Z$ given by $(z_1, z_2) \to h(z_1) - h(z_2)$ as h varies over $f^*(\Gamma(V, O_V))$, and we also note that D_Z is contained in $\overline{D}_Z$. Moreover, D_Z and $\overline{D}_Z$ are $G \times G$-stable subsets of $Z \times Z$, so to show that D_Z is dense in $\overline{D}_Z$ we can establish the corresponding fact about their images in $W \times W$. The image of D_Z is Δ_W, and the image of $\overline{D}_Z$ is $f^{-1}(\Delta_V)$. To show that Δ_W is dense in $f^{-1}(\Delta_V)$, we show that for each pair (i, j), the closure in $W_i \times W_j$ of $\Delta_W \cap (W_i \times W_j)$ is $f^{-1}(\Delta_V) \cap (W_i \times W_j)$. Let $A_i = \Gamma(W_i, O_W)$, let $A_j = \Gamma(W_j, O_W)$ and let $A_{ij} = \Gamma(W_i \cap W_j, O_W)$. The inclusion $\Delta_W \cap (W_i \times W_j)$ induces the ring map $A_i \otimes_k A_j \to A_{ij}$ which sends $a \otimes b$ to $(a|W_i \cap W_j)(b|W_i \cap W_j)$. Let I be the kernel of this ring map. The common zeros of I in $W_i \times W_j$ is the closure of $\Delta_W \cap (W_i \times W_j)$. Suppose $\sum a_r \otimes e_r \in I$, $a_r \in A_i$, $e_r \in A_j$. We can choose a common denominator d for the e_r so $e_r = c_r/d$ with $c_r \in f^*(\Gamma(V, O_V))$ and $d = (f^* b_j)^s$, where b_j is as above. Since $\sum a_r c_r = 0$,

$$\sum a_r \otimes e_r = \sum a_r \otimes e_r - \left(\sum a_r c_r\right) \otimes \frac{1}{d} = \sum (a_r \otimes 1)\left(1 \otimes e_r - c_r \otimes \frac{1}{d}\right).$$

If (x, y) is in $f^{-1}(\Delta_V) \cap (W_i \times W_j)$, then $d(y) \neq 0$ and $c_r(x) = c_r(y)$, so $e_r(y) = c_r(x)/d(y)$ and

$$\left(1 \otimes e_r - c_r \otimes \frac{1}{d}\right)(x, y) = 0.$$

It follows that $f^{-1}(\Delta_V) \cap (W_i \times W_j)$ is contained in the common zeros of I, namely the closure of $\Delta_W \cap (W_i \times W_j)$. The reverse inclusion follows from the fact that $f^{-1}(\Delta_V)$ is closed. This completes the proof of the theorem.

COROLLARY. *Let X be a normal quasi-affine variety with nontrivial G_a action. Then X/G_a exists if and only if the action is quasi-finite and locally separated.*

PROOF. If X/G_a exists, the action is necessarily quasi-finite, as noted above, and the inverse images in X of an affine open cover of X/G_a provides a cover of X by G_a-stable open sets on which the action is separated. The other implication is the theorem.

We close with the following remarks.

REMARK 1. The above theorem implies the result, mentioned above, that a proper G_a-action on a quasi-affine variety X has a quotient [**1**, Theorem 4, p. 725]: since Ψ_X is proper, its image D_X is closed, so the action is separated, and since Ψ_X is proper between quasi-affines, it has finite fibres, so the action is quasi-finite. Thus a quotient exists.

REMARK 2. Example 1 of [1, p. 727] provides an example of a quasi-finite G_a action for which no quotient exists, hence, by the theorem, is an example of a quasi-finite nonlocally separated action. Example 2 of [2] proves an example of a nonproper G_a-action on a nonsingular affine with an affine quotient, and hence is an example of a quasi-finite separated action that is not proper.

REMARK 3. The result has an analogue for the multiplicative group G_m: let X be a quasi-affine variety with G_m action. We can choose a G_m-equivariant open immersion of X into an affine variety V with G_m-action. The complement of X in V is given by the zeros of a G_m-stable ideal I of the coordinate ring of V, and hence I is generated by semi-invariants. The complements of the zeros of these semi-invariants gives an open cover of X by G_m-stable open affines. For x in X, the orbit of x is $(X \times \{x\}) \cap D_X$. Thus if the action of G_m on X is separated, D_X is closed so the orbits on X are closed and hence a geometric quotient X/G_m exists by [3, 1.3, p. 30].

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