GROUP ACTIONS ON Q-F-RINGS

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Abstract. Let $B$ be a ring, $G$ a finite group of automorphisms acting on $B$ and $B^G$ the fixed subring of $B$. We give an example of a $B$ which is quasi-Frobenius (Q-F) such that $B^G$ is not quasi-Frobenius.

S. Jøndrup [3] claims that if card $G$ is a unit in $B$ and if $B$ is self-injective then $B^G$ is self-injective. J. Fisher and J. Osterburg [2] use this assertion to prove that if $B$ is quasi-Frobenius then $B^G$ is quasi-Frobenius. However we show that this result fails.

Suppose that $A$ is a commutative local artinian ring with Jacobson radical $R$ and call $E$ the injective hull of the simple $A$-module $S = A/R$. $E$ is finitely generated and faithful [5, théorème 2, p. 97 and corollaire 6, p. 99]. $B$ denotes the ring constructed on the abelian group $A \times E$ with the multiplication

$$(a, e)(a', e') = (aa', ae' + a'e).$$

Lemma. $B$ is a commutative local quasi-Frobenius ring.

It is clear that $B$ is commutative local artinian with radical $R \times E$ and that $S' = O \times S$ is a simple ideal of $B$ essential in the ideal $O \times E$. $O \times E$ is essential in $B$, since for each $a \in A - \{0\}$ there exists $e' \in E$ such that $(o, e')(a, e) = (o, ae')$ is nonzero, because $E$ is faithful. Thus $S'$ is essential in $B$.

Then for each $f \in \text{Hom}_B(S', B) - \{0\}$, $f(S')$ and $S'$ are equal so that $\text{Hom}_B(S', B)$ is isomorphic to $S'$. By [1, Theorem 58.6, p. 396], $B$ is quasi-Frobenius. □

Proposition. (1) There exists a quasi-Frobenius ring $B$ and a finite group $G$ of automorphisms of $B$ with card $G$ invertible in $B$ such that $B^G$ is not quasi-Frobenius.

(2) It gives also an example of a quasi-Frobenius ring $C$ with an idempotent $e$ such that $eCe$ is not quasi-Frobenius.

(1) Consider a field $K$ of characteristic different from 2 and define on $K^3$ a ring $A$ by the multiplication

$$(a, b, c)(a', b', c') = (aa', ab' + a'b, ac' + a'c).$$

$A$ is a commutative local artinian ring whose socle has length 2. Thus $A$ is not quasi-Frobenius.
Define $B$ as before, $(a, e) \mapsto (a, -e)$ is an automorphism of $B$ of order 2 with fixed subring $A$.

(2) As in [2] consider the “twisted” group ring $C$ defined on the set of all formal sums $\sum_{g \in G} b_g g$ with $b_g \in B$, by the multiplication $g \cdot r = r^g g$; $e = (\sum_{g \in G} g)/\text{card } G$ is an idempotent of $C$ and $eCe$ is isomorphic to $B^G = A$ [2]. But it is easy to adapt the classical demonstration of injectivity of group rings [5, pp. 103–104] to prove that $C$ is quasi-Frobenius. □

REFERENCES


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