A REMARK ON SCHUR INDICES OF $p$-GROUPS

TOSHIHIKO YAMADA

Abstract. By making use of Hasse's sum theorem, a simple proof of the following theorem on Schur indices of $p$-groups is given.

Theorem (Roquette [3] and Solomon [4]). Let $p$ be a prime number, $G$ a $p$-group, and $\chi$ an irreducible complex character of $G$. Let $m_Q(\chi)$ denote the Schur index of $\chi$ over the rational field $Q$. Then, $m_Q(\chi) = 1$ for $p \neq 2$, and $m_Q(\chi) = 1$ or $2$ for $p = 2$.

Proof. Let $A$ be the simple component of the group algebra $Q[G]$, which corresponds to $\chi$. The center of $A$ is $Q(\chi)$, the extension field of $Q$ generated by the elements $\{\chi(g); g \in G\}$. Put $k = Q(\chi)$. Let $q$ be a rational prime (possibly the infinite prime $\infty$) and $q$ a prime of $k$, lying above $q$. Let $\text{inv}_q(\chi)$ denote the Hasse invariant of $A$ at $q$. It is well known that if $q \neq p, \infty$, then $\text{inv}_q(\chi) \equiv 0 \pmod{1}$, i.e., the Schur index $m_Q(\chi) = 1$, $Q_q$ being the $q$-adic numbers. (A result which may be established by means of modular representation theory for the prime $q \mid |G|$.)

Let $|G| = p^n$ and $\zeta$ a primitive $p^n$th root of unity. Then $k = Q(\chi) \subset Q(\xi)$. Hence there is only one prime $\nu$ of $k$ lying above $p$ (cf. Theorem 1 of [2, p. 73]). Let $\nu_{0,1}, \ldots, \nu_{\infty,n}$ be the infinite primes of $k$. Hasse's sum theorem (Satz 9, p. 119 of [1]) now yields that

$$\sum_{i=1}^{s} \text{inv}_{\nu_{i,1}}(\chi) + 2 \text{inv}_{\nu_{\infty,n}}(\chi) \equiv 0 \pmod{1}.$$  

Since $\text{inv}_{\nu_{\infty,n}}(\chi) \equiv 0$ or $\frac{1}{2}$, it follows that $\sum_{i=1}^{s} \text{inv}_{\nu_{i,1}}(\chi) \equiv 0$ or $\frac{1}{2}$ (mod 1), and consequently $\text{inv}_{\nu}(\chi) \equiv 0$ or $\frac{1}{2}$ (mod 1). This implies that $m_Q(\chi) = 1$ or $2$. Since $m_Q(\chi) \mid p^n$, then $m_Q(\chi) = 1$ for $p \neq 2$.

References


Department of Mathematics, Tokyo Metropolitan University, Fukazawa, Setagaya, Tokyo 158, Japan

Received by the editors September 21, 1977.

AMS (MOS) subject classifications (1970). Primary 20C15; Secondary 20C05.

© 1979 American Mathematical Society 0002-9939/79/0000-0361/$01.25