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AN OSCILLATION THEOREM FOR CHARACTERISTIC INITIAL VALUE PROBLEMS FOR NONLINEAR HYPERBOLIC EQUATIONS

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ABSTRACT. The nonlinear hyperbolic operator $L[u] \equiv u_{xy} + c(x, y, u)$ is studied and sufficient conditions are given that all solutions of the characteristic initial value problem for $uL[u] < 0$ are oscillatory in $(0, \infty) \times (0, \infty)$.

Recently there has been an increasing interest in studying the oscillatory behavior of solutions of characteristic initial value problems for hyperbolic equations. We refer the reader to Kreith [1] and Pagan [5], [6], in which oscillation theorems for linear equations have been obtained.

The purpose of this paper is to show that the techniques of Pagan [5] can be modified to establish oscillation criteria for the nonlinear hyperbolic operator L defined by

$$L[u] \equiv u_{xy} + c(x, y, u)$$

for

$$(x, y) \in Q_\rho \equiv \{(x, y) \in R^2 \colon 0 < x, y < \infty, \rho < y + x < \infty\},$$

where ρ is some nonnegative number. It is assumed that $c(x, y, u)$ is real-valued and continuous in $Q_\rho \times R^1$. The domain $D_L(Q_\rho)$ of L is defined to be the set of all real-valued functions of class $C^2(Q_\rho) \cap C^1(\overline{Q_\rho})$.

A function $u \in D_L(Q_\rho)$ is said to be oscillatory in Q_ρ if it has a zero in $Q(r)$ for any $r > 0$, where

$$Q(r) \equiv Q_\rho \cap \{(x, y) \in R^2 \colon (x^2 + y^2)^{1/2} > r\}.$$

We note that

$$Q(r) = \{(x, y) \in R^2 \colon 0 < x, y < \infty, (x^2 + y^2)^{1/2} > r\}$$

for sufficiently large r and $Q_0 = (0, \infty) \times (0, \infty)$.

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The following notation will be used:

$$Q(t_1, t_2) = \{(x, y) \in R^2 : 0 < x, y < \infty, t_1 < y + x < t_2\},$$

$$\Gamma_x = \{(x, 0) \in R^2 : 0 < x < \infty\},$$

$$\Gamma_y = \{(0, y) \in R^2 : 0 < y < \infty\}.$$

Associated with every function $u \in D_L(Q_\rho)$, we define the function $z(t)$ by

$$z(t) = \frac{1}{t}\int_0^t u(t - y, y)\, dy, \qquad t > \rho. \tag{1}$$

LEMMA 1. *If $u \in D_L(Q_\rho)$, then u satisfies the identity*

$$\frac{1}{t}\frac{d}{dt}\left(t^2 \frac{dz}{dt}\right) = u_x(t, 0) + u_y(0, t) + \int_0^t u_{xy}(t - y, y)\, dy, \qquad t > \rho, \tag{2}$$

where z is the function given by (1).

PROOF. Application of Green's formula yields

$$\iint_{Q(\rho,t)} u_{xy}\, dxdy = \oint_{\partial Q(\rho,t)} u_y\, dy, \tag{3}$$

where the right-hand side represents a line integral taken in the counter-clockwise direction. We easily see that the following identities hold:

$$\iint_{Q(\rho,t)} u_{xy}\, dxdy = \int_\rho^t d\tau \int_0^\tau u_{xy}(\tau - y, y)\, dy, \tag{4}$$

$$\oint_{\partial Q(\rho,t)} u_y\, dy = \int_0^t u_y(t - y, y)\, dy - \int_\rho^t u_y(0, y)\, dy - \int_0^\rho u_y(\rho - y, y)\, dy. \tag{5}$$

Differentiating both sides of (4) and (5) with respect to t and using the identity (3), we have

$$\int_0^t u_{xy}(t - y, y)\, dy = \frac{d}{dt}\int_0^t u_y(t - y, y)\, dy - u_y(0, t). \tag{6}$$

An easy computation gives

$$u_y(t - y, y) = \frac{\partial}{\partial y}[u(t - y, y)] + \frac{\partial}{\partial t}[u(t - y, y)], \tag{7}$$

so that

$$\int_0^t u_y(t - y, y)\, dy = u(0, t) - u(t, 0) + \int_0^t \frac{\partial}{\partial t}[u(t - y, y)]\, dy. \tag{8}$$

It is readily seen that

$$\int_0^t \frac{\partial}{\partial t}[u(t - y, y)]\, dy = \frac{d}{dt}\int_0^t u(t - y, y)\, dy - u(0, t), \tag{9}$$

$$\frac{d}{dt}\int_0^t u(t - y, y)\, dy = z + t\frac{dz}{dt}. \tag{10}$$

From (8)–(10) we obtain

$$\int_0^t u_y(t-y,y)\,dy = z + t\frac{dz}{dt} - u(t,0), \tag{11}$$

and therefore

$$\begin{aligned}\frac{d}{dt}\int_0^t u_y(t-y,y)dy &= \frac{d}{dt}\left(z + t\frac{dz}{dt}\right) - u_x(t,0)\\ &= \frac{1}{t}\frac{d}{dt}\left(t^2\frac{dz}{dt}\right) - u_x(t,0).\end{aligned} \tag{12}$$

Combining (6) with (12) yields the desired identity (2).

LEMMA 2. *Assume that*:

(i) $c(x,y,\xi) \geqslant p(y+x)\phi(\xi)$ *for all* $(x,y,\xi) \in Q_\rho \times (0,\infty)$, *where* p *is continuous and nonnegative in* (ρ,∞) *and* ϕ *is continuous, nonnegative and convex in* $(0,\infty)$,

(ii) $u \in D_L(Q_\rho)$ *is a positive function such that*

$$\begin{aligned} L[u] &\leqslant 0 \quad \text{in } Q_\rho,\\ u_x &\leqslant 0 \quad \text{on } \Gamma_x \cap \{x > \rho\},\\ u_y &\leqslant 0 \quad \text{on } \Gamma_y \cap \{y > \rho\}.\end{aligned}$$

Then the function $z(t)$ *given by* (1) *satisfies the ordinary differential inequality*

$$\frac{d}{dt}\left(t^2\frac{dz}{dt}\right) + t^2p(t)\phi(z) \leqslant 0, \qquad \rho < t < \infty. \tag{13}$$

PROOF. By the hypotheses (i) and (ii) we get

$$\begin{aligned}\int_0^t u_{xy}(t-y,y)\,dy &\leqslant -\int_0^t c(t-y,y,u(t-y,y))\,dy\\ &\leqslant -p(t)\int_0^t \phi(u(t-y,y))\,dy.\end{aligned} \tag{14}$$

Applying Jensen's inequality [4, p. 160], we obtain

$$\int_0^t \phi(u(t-y,y))\,dy \geqslant t\phi(z). \tag{15}$$

It follows from (14) and (15) that

$$\int_0^t u_{xy}(t-y,y)\,dy \leqslant -tp(t)\phi(z). \tag{16}$$

We observe, using (2) and (16), that

$$\frac{1}{t}\frac{d}{dt}\left(t^2\frac{dz}{dt}\right) + tp(t)\phi(z) \leqslant u_x(t,0) + u_y(0,t). \tag{17}$$

Since the right side of (17) is nonpositive by the hypothesis (ii), we obtain the desired inequality (13).

THEOREM 3. *Assume that the following conditions are satisfied*:

(i) *the hypothesis* (i) *of Lemma* 2,

(ii) $c(x, y, -\xi) = -c(x, y, \xi)$ *for all* $(x, y, \xi) \in Q_\rho \times (0, \infty)$,

(iii) *the ordinary differential inequality* (13) *has no positive solution in* $[t, \infty)$ *for any* $t > \rho$.

Then every solution $u \in D_L(Q_\rho)$ *of the characteristic initial value problem*

$$uL[u] \leqslant 0 \quad \text{in } Q_\rho, \tag{18}$$

$$u_x + \lambda_1(x)u = 0 \quad \text{on } \Gamma_x \cap \{x > \rho\}, \tag{19}$$

$$u_y + \lambda_2(y)u = 0 \quad \text{on } \Gamma_y \cap \{y > \rho\}, \tag{20}$$

is oscillatory in Q_ρ, *where* λ_i $(i = 1, 2)$ *are continuous functions on* $\Gamma_x \cap \{x > \rho\}$ *and* $\Gamma_y \cap \{y > \rho\}$ *such that* $0 \leqslant \lambda_i < \infty$, *respectively*.

PROOF. Suppose to the contrary that there exists a solution u of the problem (18)–(20) which has no zero in $Q(r_0)$ for some $r_0 > 0$. Without loss of generality we may assume that

$$u > 0 \quad \text{in } Q(r_0),$$
$$L[u] \leqslant 0 \quad \text{in } Q(r_0),$$
$$u_x = -\lambda_1(x)u \leqslant 0 \quad \text{on } \Gamma_x \cap \{x > r_0\},$$
$$u_y = -\lambda_2(y)u \leqslant 0 \quad \text{on } \Gamma_y \cap \{y > r_0\}.$$

There exists a positive number t_0 such that $Q(t_0, \infty) \subset Q(r_0)$. Then, we see from Lemma 2 that the function $z(t)$ given by (1) is a positive solution of (13) in (t_0, ∞). This contradicts the hypothesis (iii) and completes the proof.

Sufficient conditions for the nonexistence of an eventually positive solution of (13) have been obtained by Naito and Yoshida [2] and Noussair and Swanson [3]. Theorem 3 combined with the results of [2], [3] yield the following corollary.

COROLLARY 4. *Let* $\gamma \geqslant 1$ *be the quotient of odd integers. Every solution* $u \in D_L(Q_\rho)$ *of the characteristic initial value problem*

$$u_{xy} + c(x, y)u^\gamma = 0 \quad \text{in } Q_\rho,$$

$$u_x + \lambda_1(x)u = 0 \quad \text{on } \Gamma_x \cap \{x > \rho\}, \qquad 0 \leqslant \lambda_1(x) < \infty, \tag{21}$$

$$u_y + \lambda_2(y)u = 0 \quad \text{on } \Gamma_y \cap \{y > \rho\}, \qquad 0 \leqslant \lambda_2(y) < \infty, \tag{22}$$

is oscillatory in Q_ρ *if the following conditions are satisfied*:

(i) $c(x, y) \geqslant p(y + x)$ *in* Q_ρ, *where* p *is continuous and nonnegative in* (ρ, ∞),

(ii) $\int^\infty \psi_\gamma(t)p(t)dt = \infty$, *where*

$$\psi_\gamma(t) = \begin{cases} t^{2-\gamma}, & \gamma > 1, \\ t^2\left\{[l_m(t)]^\mu \prod_{i=0}^{m} l_i(t)\right\}^{-1}, & \gamma = 1, \end{cases}$$

for some number $\mu > 0$ *and some integer* $m \geqslant 0$, *where* $l_0(t) = t$ *and* $l_i(t) = \log(l_{i-1}(t) + 1)$.

COROLLARY 5. *If* γ $(1 \leqslant \gamma \leqslant 3)$ *is the quotient of odd integers, then every solution* $u \in D_L(Q_\rho)$ *of the equation*

$$u_{xy} + k^2 u^\gamma = 0 \quad \text{in } Q_\rho, \tag{23}$$

which satisfies the boundary conditions (21) *and* (22), *is oscillatory in* Q_ρ, *where* k *is a positive constant.*

PROOF. Since

$$\int^\infty t^2\{t^{1/2}t\}^{-1}k^2\,dt = k^2\int^\infty t^{1/2}\,dt = \infty,$$

$$\int^\infty t^{2-\gamma}k^2\,dt = \infty, \qquad 1 < \gamma \leqslant 3,$$

the conclusion follows from Corollary 4.

In the linear case $\gamma = 1$, the oscillation results of this paper are closely related to those obtained by Pagan [5].

As is well known, the characteristic initial value problem

$$u_{xy} + k^2 u = 0 \quad \text{in } Q_0,$$
$$u = 1 \quad \text{on } \Gamma_x \cup \Gamma_y,$$

has an oscillatory solution $J_0(2k(xy)^{1/2})$, where J_0 is the Bessel function of the first kind of order 0 (see, e.g., Kreith [1]). We find that $J_0(2k(xy)^{1/2})$ is also an oscillatory solution of the problem (21)–(23) with $\gamma = 1$, $\lambda_1(x) \equiv 0$ and $\lambda_2(y) \equiv 0$.

We conclude by observing that Corollary 4 becomes false if either (21) or (22) is omitted. We limit ourselves to the case where (21) is omitted, because the case where (22) is omitted can be discussed analogously. Consider the problem

$$u_{xy} + \tfrac{1}{4}\big((y+x)^{-7/3} + 6(y+x)^{-4/3}\big)e^{2y}u^{5/3} = 0 \quad \text{in } Q_\rho, \tag{24}$$

$$u_y + \big(3 - (2y)^{-1}\big)u = 0 \quad \text{on } \Gamma_y \cap \{y > \rho\}, \tag{25}$$

where $\rho > \frac{1}{6}$. Here $\gamma = \frac{5}{3}$ and $\lambda_2(y) = 3 - (2y)^{-1} \geqslant 0$. Since $e^{2y} \geqslant 1$ in Q_ρ and

$$\int^\infty \psi_{5/3}(t)p(t)dt = \int^\infty t^{1/3}\big(\tfrac{1}{4}\big)\big(t^{-7/3} + 6t^{-4/3}\big)\,dt = \infty,$$

conditions (i) and (ii) of Corollary 4 are satisfied. Equation (24) has a nonoscillatory solution $v \equiv (y+x)^{1/2}e^{-3y}$ which satisfies the boundary condition (25). However, v does not satisfy the boundary contition (21). In fact, we obtain

$$v_x + \lambda_1(x)v = \tfrac{1}{2}x^{-1/2}(1 + 2\lambda_1(x)x) > 0 \quad \text{on } \Gamma_x \cap \{x > \rho\}$$

for any $\lambda_1(x)$ $(0 \leqslant \lambda_1(x) < \infty)$.

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