PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 1, August 1979

POINCARÉ DUALITY AND FIBRATIONS

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ABSTRACT. Let $F \to E \to B$ be a fibration such that F, E, and B are homotopy equivalent to finite complexes. Then the following fact is proved. E is a Poincaré duality space if and only if B and F are Poincaré duality spaces.

Let $F \to E \to B$ be a fibration with F, B and hence E being homotopy equivalent to finite complexes. We shall give a proof of the following theorem.

THEOREM 1. *E is a Poincaré duality space if and only if B and F are Poincaré duality spaces. Here Poincaré duality space is as defined in* [W].

This theorem was announced by F. Quinn in [Q], but up to now no proof has appeared in the literature; R. Schultz and R. Rigdon have also discovered proofs. The result has been used in [CG] and by R. Schultz. So a brief proof in the literature has become desirable.

Suppose X is a finite complex. We may embed X in R^n so that there exists a regular neighborhood N of X which has X as a deformation retract. Then the inclusion $\partial N \to N \simeq X$ has a homotopy theoretical fibre denoted Φ_X. We know from work of Spivak, [S], that X satisfies Poincaré duality for twisted coefficients over $\pi_1(X)$ if and only if Φ_X is homotopy equivalent to a sphere.

We shall show that Φ_E is homotopy equivalent to the join $\Phi_B * \Phi_F$. (We may embed F, E and B in large enough Euclidean spaces to insure that Φ_E, Φ_B and Φ_F are simply connected.) Then by the Künneth formula Φ_E is a sphere if and only if Φ_F and Φ_B are spheres. In view of Spivak's result, this will establish the theorem.

We shall prove the theorem for the special case of a locally trivial fibre bundle in which the fibre, total space and base are manifolds with boundary. This will establish the general case for Hurewicz fibrations since by the closed fibre smoothing theorem of [CG] we know that for any fibration $F \to E \to B$ where B is finite dimensional and F is a finite complex, there is a torus T^n and a locally trivial smooth bundle $N \to M \to B$ such that $N \to M \to B$ is fibre homotopy equivalent to $F \times T^n \xrightarrow{i \times 1} E \times T^n \to B$. Thus if $\Phi_M \simeq \Phi_N * \Phi_B$ then $\Phi_{E \times T^n} \simeq \Phi_{F \times T^n} * \Phi_B$. But $\Phi_{E \times T^n} \simeq \Phi_E * \Phi_{T^n} \simeq \Phi_E * S^m$, so

Received by the editors September 29, 1978.
AMS (MOS) subject classifications (1970). Primary 55F05, 57B10.
[1]This work was partially supported by a grant from the National Science Foundation.

$\Phi_E * S^k \simeq \Phi_F * \Phi_B * S^l$. Hence the theorem for fibrations will follow from the special case of fibre bundles.

So assume that $F \to E \to B$ is a locally trivial fibre bundle for which F, E and B are manifolds with boundaries. Now if N is a manifold with boundary ∂N, we convert the inclusion $\partial N \subset N$ into a fibration by letting $\bar{E}$ be the space of paths σ in N such that $\sigma(0) \in \partial N$. The projection p: $\bar{E} \to N$ is given by $p(\sigma) = \sigma(1)$ and the homotopy theoretic fibre Φ_N over the base point $n_0 \in N$ is $\Phi_N = \{\sigma \in N^I | \sigma(0) \in \partial N \text{ and } \sigma(1) = n_0\}$.

Now let $e_0 \in E$ be a base point and let $P(E)$, $P(F)$, and $P(B)$ be the space of paths in E, F, and B, respectively, which end at e_0, e_0, and $\pi(e_0)$, respectively, where $E \xrightarrow{\pi} B$ is the projection. Observe that $\Phi_F \subset P(F)$ and $\Phi_B \subset P(B)$ and $\Phi_E \subset P(E)$. We shall prove that $P(E)$ is homeomorphic to $P(F) \times P(B)$. Thus $\Phi_E = (\Phi_B \times P(F)) \cup (P(B) \times \Phi_F)$.

LEMMA. *$P(E) = P(B) \times P(F)$ and $\Phi_E = (\Phi_B \times P(F)) \cup (P(B) \times \Phi_F)$.*

PROOF. For any fibration $F \to E \xrightarrow{\pi} B$ there is a lifting function λ: $\bar{B} \to E^I$ where $\bar{B}$ is the subspace of $E \times B^I$ given by $\{(e, \sigma) | \pi(e) = \sigma(0)\}$ and $\lambda(e, \sigma)$ is a path $\tilde{\sigma} \in E^I$ such that $\pi \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = e$. So let λ be the lifting function for the principle fibre bundle $G \to E^{(F)} \to B$ associated to $F \to E \to B$. Here G is the group of homeomorphisms of F and $E^{(F)}$ is the space of maps of the distinguished fibre F into any other fibre which is a homeomorphism onto that fibre. If we let i: $F \to E$ be the obvious inclusion we see that every path $\sigma \in P(B)$ gives rise to an isotopy of homeomorphisms $\tilde{\sigma}_t$: $F \to E$ such that $\pi \circ \tilde{\sigma}_t(e_0) = \sigma_t(t)$. Note $t \mapsto \tilde{\sigma}_t(e_0)$ defines a lifting $\tilde{\sigma}$ of σ. Now we define a continuous map γ: $P(F) \times P(B) \to P(E)$ by $(\tau, \sigma) \mapsto \rho$ where $\rho(t) = \tilde{\sigma}_t(\tau(t))$. Also define δ: $P(E) \to P(F) \times P(B)$ by $\rho \mapsto (\tau, \sigma)$ where $\sigma = \pi \circ \rho$ and $\tau(t) = \tilde{\sigma}_t^{-1}(\rho(t))$. Now ρ and δ are inverse to each other so $P(E)$ is homeomorphic to $P(B) \times P(F)$.

To see that $\Phi_E = (\Phi_B \times P(F)) \cup (P(B) \times \Phi_F)$ we need only apply the construction to the boundary ∂E which is the union of $p^{-1}(\partial B)$ and the subbundle of E given by all the points which lie in the boundary of some fibre of E.

Now the pairs $(P(E), \Phi_E)$, $(P(F), \Phi_F)$ and $(P(B), \Phi_B)$ are homotopy equivalent to the pairs $(C\Phi_E, \Phi_E)$, $(C\Phi_F, \Phi_F)$ and $(C\Phi_B, \Phi_B)$, respectively, where $C\Phi_E$ is the cone on Φ_E, etc. This follows since the space $P(E)$ is contractible and the pair $(P(E), \Phi_E)$ has the homotopy extension property since ∂E has a collar neighborhood in E. Thus $(P(E), \Phi_E)$ is homotopy equivalent to $(C(\Phi_F) \times C(\Phi_B), \Phi_F * \Phi_B)$ so Φ_E is homotopy equivalent to $\Phi_F * \Phi_B$.

BIBLIOGRAPHY

[CG] A. Casson and Daniel H. Gottlieb, *Fibrations with compact fibres*, Amer. J. Math. **99** (1977), 159–189.

[S] M. Spivak, *Spaces satisfying Poincaré duality*, Topology **6** (1967), 77–102.

[Q] F. Quinn, *Surgery on Poincaré and normal spaces*, Bull. Amer. Math. Soc. **78** (1972), 262–267.

[W] C. T. C. Wall, *Surgery of non-simply-connected manifolds*, Ann. of Math. (2) **84** (1966), 217–276.

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