DEMENSION AND MEASURE

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ABSTRACT. We give a new characterization, based on Hausdorff measure, for the dimension of a compact set in a euclidean space.

The demension $\text{dem} \ X$ (dimension of embedding) of a compact set $X$ in a euclidean space $\mathbb{R}^n$ was introduced by Štan’ko [4] to characterize the tameness of $X$ in $\mathbb{R}^n$. A good exposition of the theory has been given by Edwards [1]. There are several equivalent definitions for $\text{dem} \ X$ (see [1, §1.2] and Štan’ko’s original definition [4, §1.1]). In this note we give a measure-theoretic characterization of $\text{dem} \ X$, which is an analogue of the theorem of Szpilrajn (= Marczewski) concerning $\text{dim} \ X$ [2, Theorem VII 1, p. 102].

We let $m_\alpha(X)$ denote the $\alpha$-dimensional Hausdorff measure of a set $X \subseteq \mathbb{R}^n$ and $\text{dim}_H X$ the Hausdorff dimension of $X$. For definitions, see [2, pp. 103, 107].

THEOREM. Let $X$ be a compact set in $\mathbb{R}^n$. Then $\text{dem} \ X < k$ if and only if there is a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $m_{k+1}(fX) = 0$. Moreover, $\text{dem} \ X \leq \text{dim}_H fX$ for all homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$, and $\text{dem} \ X = \text{dim}_H fX$ for some $f$.

Proof. If $m_{k+1}(fX) = 0$ for some homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, then $\text{dem} fX < k$ by [3, 6.15]. Since $\text{dem} X$ is invariant under homeomorphisms of $\mathbb{R}^n$ [1, §1.1], this implies $\text{dem} X < k$.

To complete the proof of the theorem, it suffices to construct a compact set $P^k_n \subseteq \mathbb{R}^n$ such that $\text{dim}_H P^k_n < k$ and such that for every compact set $X \subseteq \mathbb{R}^n$ with $\text{dem} X < k$ there is a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ which maps $X$ into $P^k_n$. We shall construct $P^k_n$ by modifying the construction of Menger’s compactum $M^k_n$.

We start with the unit cube $I^n = [0, 1]^n$. Subdivide $I^n$ into $4^n$ cubes of side length $\frac{1}{4}$ and retain those which meet the $k$-faces of $I^n$. These will be called cubes of rank one. Proceeding inductively, assume that $Q$ is a cube of rank $j - 1$. Subdivide $Q$ into $2^{(j+1)n}$ equal cubes. Those which meet the $k$-faces of $Q$ are called cubes of rank $j$. Let $S_j$ be the union of all cubes of rank $j$. Then $P^k_n = \bigcap \{S_j | j > 1\}$.

We next show that $\text{dim}_H P^k_n < k$, that is, $m_\alpha(P^k_n) = 0$ for every $\alpha > k$. Let $r$ be the number of all $k$-faces of $I^n$. Since each cube of rank $j - 1$ contains at most $2^{(j+1)k}r$ cubes of rank $j$, there are at most $2^k \times 2 \times 2 \times \cdots \times 2 \times 2 \times 2$ cubes of rank $j$. Hence, $m_\alpha(P^k_n) = 0$ for every $\alpha > k$.
cubes of rank \( j \). The side length of such a cube is \( 2^{-j(\alpha+3)/2} \). Hence these cubes form a cover \( \{ Q_1, \ldots, Q_s \} \) of \( P_n^k \) such that
\[
\sum_{i=1}^{s} d(Q_i)^{\alpha} \leq \left( 2^{(k-\alpha)(\alpha+3)/2} r \right) \gamma_n^{\alpha/2}.
\]
Since \( \alpha > k \), the right-hand side tends to zero as \( j \rightarrow \infty \). Thus \( m_{\alpha}(P_n^k) = 0 \).

Suppose that \( X \) is compact in \( \mathbb{R}^n \) and \( \text{dem} \ X < k \). By a result of Štan’ko [5] (see also Edwards [1, §1.2]), there is a homeomorphism \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) (in fact, an isotopy of \( \mathbb{R}^n \) with compact support) which carries \( X \) into Menger’s compactum \( M_n^k \). It is easy to see that \( M_n^k \) can be replaced by \( P_n^k \) in the proof of this result. □

Remarks. There is an isotopy version of the above result, since the map \( f \) in the proof can be obtained by an isotopy of \( \mathbb{R}^n \) with compact support. In fact, the isotopy can be chosen to be arbitrarily small by using a stack of small copies of \( P_n^k \) (cf. Edwards [1, pp. 208–209]).

The result can be extended to closed subsets of Lipschitz manifolds (cf. [1, §2]).

References