MAJORIZATION ON A PARTIALLY ORDERED SET

F. K. HWANG

Abstract. We extend the classical concept of set majorization to the case where the set is partially ordered. We give a useful property which characterizes majorization on a partially ordered set. Quite unexpectedly, the proof of this property relies on a theorem of Shapley on convex games. We also give a theorem which is parallel to the Schur-Ostrowski theorem in comparing two sets of parameters in a function.

1. Introduction. The classical concept of majorization is defined on two n-sets of numbers \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_n\} \) as follows. Let \( a_{[i]} \) and \( b_{[i]} \) denote the \( i \)th largest numbers in \( A \) and \( B \), respectively. Then \( A \) is said to majorize \( B \) if and only if

\[
\sum_{i=1}^{k} a_{[i]} > \sum_{i=1}^{k} b_{[i]} \quad \text{for } k = 1, \ldots, n - 1
\]

and

\[
\sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}.
\]

The concept of majorization is closely related to the concept of a Schur function. A function \( f(x_1, \ldots, x_n) \) is called a Schur function \([4]\) if, for all \( i \) and \( j \),

\[
\left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right) (x_i - x_j) > 0.
\]

The following theorem connects the two concepts:

Theorem 1.1 (Schur [5], Ostrowski [4]). \( f(a_1, \ldots, a_n) > f(b_1, \ldots, b_n) \) for all \( A \) majorizing \( B \) if and only if \( f \) is a Schur function.

Set majorization can be naturally extended to vector majorization. We say that an \( n \)-vector \( A = (a_1, \ldots, a_n) \) majorizes another \( n \)-vector \( B = (b_1, \ldots, b_n) \) if and only if

\[
\sum_{i=1}^{k} a_i > \sum_{i=1}^{k} b_i \quad \text{for } k = 1, \ldots, n - 1,
\]

and

\[
\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.
\]
The concept of vector majorization has been proved useful in many instances ([1], [3]) where vector optimization is concerned.

Vector majorization can be interpreted in a different way which leads to a further extension. Let \( P = \{ p_1, \ldots, p_n \} \) be a set of points ordered linearly by \( \rightarrow \) and let \( A = \{ a_1, \ldots, a_n \} \) and \( B = \{ b_1, \ldots, b_n \} \) be two sets of weights where \( a_i \) and \( b_i \) are associated with \( p_i \) for \( i = 1, \ldots, n \). Then \( A \) is said to majorize \( B \) if for every point \( p_i \) in \( P \), the sum of the \( a \) weights of all points \( \{ p_j : p_j \rightarrow p_i \text{ or } p_j = p_i \} \) is not less than the sum of the \( b \) weights on the same set of points. With this viewpoint, it is natural to consider majorization on a set of points \( P \) partially ordered by \( \rightarrow \) (read "dominates"). For \( P' \subseteq P \) let \( A(P') \) (or \( B(P') \)) denote the sum of the weights (\( b \) weights) of all the points \( \{ p_j : p_j \in P' \text{ or } p_j \rightarrow p_i \text{ for some } p_i \in P' \} \). Then we say that \( A \) majorizes \( B \) on \( P \) if \( A(P) = B(P) \) and for every \( P' \subseteq P \), \( A(P') > B(P') \). Note that when \( P \) is a linearly ordered set, then \( A \) majorizing \( B \) on \( P \) is reduced to the definition of vector majorization.

In this paper we give a useful property which characterizes majorization on a partially ordered set. It turns out that to prove this property, we need to resort to some concepts and results in characteristic function games. Therefore we give a brief sketch of what we need from characteristic function games in \( \S 2 \). Using the characterization property, we prove a theorem parallel to the Schur-Ostrowski theorem on set majorization. A similar theorem on vector majorization follows as a corollary to our theorem.

### 2. Some concepts and results in characteristic function games.

For a set of players \( N = \{ 1, \ldots, n \} \) a characteristic function \( \nu(\cdot) \) is a real valued function assigning to each subset \( S \subseteq N \) the number \( \nu(S) \). This number may be thought of as describing the potential worth of the coalition \( S \). The function \( \nu(\cdot) \) completely determines the strategic possibilities of the game. A game is convex if its characteristic function satisfies \( \nu(\phi) = 0 \) and

\[
\nu(S) + \nu(T) < \nu(S \cup T) + \nu(S \cap T)
\]

for any \( S, T \subseteq N \).

The core of a characteristic function game is the set of solutions to the following set of equations

\[
\sum_{i \in S} x_i > \nu(S) \quad \text{for } S \subseteq N,
\]

\[
\sum_{i \in N} x_i = \nu(N).
\]

The core can be described intuitively as the set of payoff vectors that leave no coalition in a position to improve the payoffs of all its members. A characteristic function game need not have a core. However, Shapley [6] proved the following.

**Theorem 2.1.** The core exists for every convex game.
It is quite unexpected that this theorem will be needed to prove a property of majorization on a partially ordered set.

3. The main theorems. Consider a given set of points $P = \{p_1, \ldots, p_n\}$ partially ordered by "$\rightarrow"$ and a set of weights $A$ associated with $P$. Let $p_i$ and $p_j$ be two points in $P$ such that $p_i \rightarrow p_j$. Then a flow from $a_i$ to $a_j$ is a transformation from $A$ to $A'$ where

$$a'_i = a_i - \delta,$$
$$a'_j = a_j + \delta,$$
$$a'_k = a_k \quad \text{for } k \neq i, j,$$

for some $\delta > 0$.

**Theorem 3.1.** Let $A$ and $B$ be two sets of weights associated with the partially ordered set $P$. Then $A$ majorizes $B$ on $P$ if and only if $A$ can be transformed into $B$ by a finite set of flows (in fact, at most $\binom{n}{2}$ flows are needed).

**Proof.** (i) The "if" direction. If $A$ can be transformed into $A'$ by a flow, then clearly $A$ majorizes $A'$ on $P$. Since majorization on $P$ is transitive, $A$ majorizes $B$ on $P$.

(ii) The "only if" direction. Suppose $A$ majorizes $B$ on $P$; we show that there exists a finite set of flows transforming $A$ into $B$. We prove this by induction on the number of points in $P$. Let $q$ be a point of $P$ which is not dominated by any other point of $P$. If $q$ does not dominate any other point we ignore $q$ and prove Theorem 3.1 by induction. Otherwise let $q_1, \ldots, q_j$ be the points dominated by $q$, but not dominated by any other points dominated by $q$. If $a_q = b_q$, then again, we can ignore $q$ and prove Theorem 3.1 by induction. So we assume $a_q - b_q = \theta > 0$. We now show that there exists a set of weights $A'$ which can be obtained from $A$ by flowing the amount $\theta_i > 0$ from $q$ to $q_i$, $i = 1, \ldots, j$, such that $\sum_{i=1}^{j} \theta_i = \theta$ and $A'$ majorizes $B$ on $P$. Once this has been proved, then by induction $A'$ can be transformed into $B$ by a finite set of flows. Consequently, $A$ can be transformed into $B$ by a finite set of flows and Theorem 3.1 follows.

For $P' \subseteq P$, define

$$\bar{A}(P') = A(P') - \theta.$$

Let $K$ be a subset of $J = \{1, \ldots, j\}$. Define

$$v_K = \max_{P_K} \left( B(P_K) - \bar{A}(P_K) \right)$$

where $P_K$ is a subset of $P - \{q\}$ containing $K$ but not any element from $J - K$. Then $A'$ majorizes $B$ if and only if

$$\sum_{i \in K} \theta_i > v_K \quad \text{for every } K \subseteq J.$$

We can define a characteristic function game on the set of players $J$ by treating $\{v_K: K \subseteq J\}$ as the characteristic function. Then
\[ \sum_{i \in K} \theta_i > v_K \quad \text{for every } K \subseteq J \]

is equivalent to the statement that the core of the game exists. We now prove the core exists by showing that the game is convex.

Let \( P^0_K \) and \( P^0'_K \) be two subsets of \( J \) such that

\[ v_K = B(P^0_K) - \overline{A}(P^0_K) \]

and

\[ v_{K'} = B(P^0_{K'}) - \overline{A}(P^0_{K'}) \]

Then

\[ v_{K \cup K'} > B(P^0_K \cup P^0_{K'}) - \overline{A}(P^0_K \cup P^0_{K'}) \]

\[ = B(P^0_K) + B(P^0_{K'}) - B(P^0_K \cap P^0_{K'}) \]

\[ - \overline{A}(P^0_K) - \overline{A}(P^0_{K'}) + \overline{A}(P^0_K \cap P^0_{K'}) \]

\[ = v_K + v_{K'} - (B(P^0_K \cap P^0_{K'}) - \overline{A}(P^0_K \cap P^0_{K'})) \]

\[ > v_K + v_{K'} - v_{K \cap K'} \]

Therefore the game is convex and so the core exists by Theorem 2.1.

**Corollary.** Suppose \( P \) is a linearly ordered set. Then a necessary and sufficient condition for \( A \) majorizing \( B \) is the existence of an \( n \times n \) triangular matrix \( M = \{m_{ij}\} \) such that \( m_{ij} > 0, m_{ij} = 0 \) for \( i < j, \sum_{j=1}^{n} m_{ij} = 1 \) and \( B = MA \) (by interpreting \( m_{ij}a_i \) as the amount of flow from \( p_i \) to \( p_j \)).

Note that this corollary is very similar to Theorem 46 of [2] which says that a necessary and sufficient condition for a set \( A \) majorizing a set \( B \) is the existence of a doubly stochastic matrix \( M \) such that \( B = MA \).

The following theorem is parallel to the Schur-Ostrowski theorem.

**Theorem 3.2.** Let \( f(x_1, \ldots, x_n) \) be a function defined over the domain \( D \). Let \( P = (p_1, \ldots, p_n) \) be a set of points partially ordered by \( \rightarrow \). Then

\[ f(a_1, \ldots, a_n) \geq f(b_1, \ldots, b_n) \]

for all \( A \) majorizing \( B \) on \( P \) if and only if \( f \) is such that for every \( i \) and \( j \), \( p_i \rightarrow p_j \) implies

\[ \frac{\partial f}{\partial x_i} \geq \frac{\partial f}{\partial x_j} \quad \text{over all } X \in D. \]

**Proof.** The "only if" part is trivial. The "if" part can be proved as follows.

If there is a flow transforming \( A \) into \( A' \), then clearly, \( f(A) \geq f(A') \). From Theorem 3.1, there exists a finite set of flows transforming \( A \) into \( B \). Therefore Theorem 3.2 follows.
Corollary. Let $f(x_1, \ldots, x_n)$ be a function defined over domain $D$. Let $P = (p_1, \ldots, p_n)$ be a vector. Then
$$f(a_1, \ldots, a_n) > f(b_1, \ldots, b_n)$$
for all $A$ majorizing $B$ (in the vector sense) if and only if $f$ is such that
$$\left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right)(i - j) > 0.$$

4. Conclusions. In this paper we extend the classical concept of set majorization to the case where the set is partially ordered. We give a mathematical definition of this new concept which includes "vector majorization" as a special case. Let $P$ and $P'$ be two partial orders on the same set such that $p_i \rightarrow p_j$ in $P$ implies $p_i \rightarrow p_j$ in $P'$. Then surprisingly, it is not true that if $A$ majorizes $B$ on the set under $P'$, then $A$ majorizes $B$ on the set under $P$ (nor conversely) as is clear from the following example:

Example. Let $P = (p_1 \rightarrow p_2, p_1 \rightarrow p_3), P' = (p_1 \rightarrow p_2 \rightarrow p_3), A = (x_1 = .5, x_2 = .5, x_3 = 0), B = (y_1 = .4, y_2 = .3, y_3 = .3)$. Then $A$ majorizes $B$ on $P'$ but not on $P$.

We also prove a property which characterizes majorization on a partially ordered set. Quite unexpectedly, the proof relies on a theorem of Shapley on convex games. Furthermore, we prove a theorem which is parallel to the Schur-Ostrowski theorem in comparing two functions except that the set majorization condition is replaced by a condition relating to the new notion of majorization.

The author wishes to thank a referee for suggesting the corollary of Theorem 3.1.

References


Bell Laboratories, Murray Hill, New Jersey 07974