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PERIODIC MODULES WITH LARGE PERIODS

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ABSTRACT. Let G be a nonabelian group of order p^3 and exponent p, where p is an odd prime. Let K be a field of characteristic p. In this paper it is proved that there exist periodic KG-modules whose periods are $2p$. Some examples of such modules are constructed.

1. Introduction. Let G be a finite p-group, and let K be a field of characteristic p, where p is a prime integer. If M is a KG-module then there exists a projective KG-module F such that there is an epimorphism φ: $F \to M$. The kernel of φ can be written as $\Omega(M) \oplus E$ where E is projective and $\Omega(M)$ has no projective submodules. It is well known [5] that the isomorphism class of $\Omega(M)$ is independent of the choice of F and φ. Inductively we define $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for all integers $n > 1$. A KG-module is said to be periodic if there exists an integer $n > 0$ such that $M \cong \Omega^n(M) \oplus P$ where P is a projective KG-module. If n is the smallest such integer then n is called the period of M.

Recently it has been proved that, when G is abelian, every periodic KG-module has period 1 or 2 (see [2] or [4]). It is well known that if G is a quaternion group, then every KG-module has period 1, 2 or 4. Until now there were no known examples of periodic modules with periods other than 1, 2, or 4 (see [1]). In this paper we show that if G is a group of order p^3 and exponent p for p an odd prime, then there exist periodic KG-modules with period $2p$. Some examples along with their minimal projective resolutions are explicitly constructed.

2. Notation and preliminaries. Let p be an odd prime integer. Suppose that K is a field of characteristic p. Throughout the rest of this paper G will denote the group of order p^3 and exponent p. Then G is generated by two elements x and y. If $z = x^{-1}y^{-1}xy$, then we have the relations $x^p = y^p = z^p = 1$, $xz = zx$, and $yz = zy$. Let H be the subgroup generated by x and z. Let

$$\tilde{H} = \sum_{h \in H} h = (x-1)^{p-1}(z-1)^{p-1} \in KH.$$

Recall that a KH-module L is free if and only if $\mathrm{Dim}_K \tilde{H}L = (1/p^2)\mathrm{Dim}_K L$. If M is a KG-module, then M_H is its restriction to a KH-module.

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The remainder of this section is devoted to establishing some combinatorial relations which will be needed in the next section. For $\alpha \in K$, let $l = l(\alpha) = (y - 1) - \alpha(x - 1) \in KG$.

LEMMA 2.1. *$l^p = k(z - 1)^{p-1}$ where*

$$k = \sum_{t=1}^{p-1} (-1)^t \frac{1}{p}\binom{p}{t}\alpha^t x^t y^{p-t}.$$

PROOF. Note that $l = (\alpha - 1) + (y - \alpha x)$. Therefore

$$l^p = (\alpha - 1)^p + (y - \alpha x^p) = \alpha^p - 1 + (y - \alpha x)^p.$$

In the expansion of $(y - \alpha x)^p$ the coefficient on $(-\alpha)^t$ is the sum of all possible products of x's and y's with x occurring t times and y occurring $p - t$ times. Each such product can be written in the form $x^t y^{p-t} z^s$ for some $s = 0, 1, \ldots, p - 1$. Suppose that in each product the left-most letter (either x or y) is moved to the furthest right position. This operation amounts to multiplying the product by z^t. However the entire sum remains unchanged. Hence for $t = 1, \ldots, p - 1$, the coefficient on $(-\alpha)^t$ is

$$\sum_{j=0}^{p-1} \frac{1}{p}\binom{p}{t} x^t y^{p-t} z^j = \frac{1}{p}\binom{p}{t} x^t y^{p-t}(z - 1)^{p-1}.$$

This proves the lemma.

LEMMA 2.2. *There exists an element $\nu \in KG$ such that $lk - kl = \nu(z - 1)$. Moreover if α is an element of K which is not in the prime field F_p, then ν is a unit in KG.*

PROOF. Now $lk - kl = (yk - ky) - \alpha(xk - kx)$. So

$$lk - kl = \sum_{t=1}^{p-1} (-1)^t \frac{1}{p}\binom{p}{t}\alpha^t x^t y^{p-t+1}(z^{-t} - 1)$$
$$- \sum_{t=1}^{p-1} (-1)^t \frac{1}{p}\binom{p}{t}\alpha^{t+1} x^{t+1} y^{p-t}(1 - z^t).$$

Note that $z^{-t} - 1 = z^{p-t} - 1 = (z - 1)(z^{p-t-1} + \cdots + z + 1)$. Hence $lk - kl = \nu(z - 1)$ where

$$\nu = \alpha x z^{-1} - \alpha^p y z^{-1}$$
$$- \sum_{t=1}^{p-2} (-1)^t \frac{1}{p}\left[\binom{p}{t+1}(z^{p-t-2} + \cdots + 1)\right.$$
$$\left. - \binom{p}{t}(z^{t-1} + \cdots + 1)\right]\alpha^{t+1} x^{t+1} y^{p-t}.$$

Let ϵ: $KG \to K$ be the augmentation homomorphism given by $\epsilon(g) = 1$ for all $g \in G$. Then

$$\epsilon(\nu) = \alpha - \alpha^p - \sum_{t=1}^{p-2} (-1)^t \left[\frac{1}{p}\binom{p}{t+1}(-t-1) - \frac{1}{p}\binom{p}{t}(t-p) \right] \alpha^{t+1}$$
$$= \alpha - \alpha^p.$$

The lemma follows from the fact that ν is a unit if and only if $\epsilon(\nu) \neq 0$.

Now if $r = 1, \ldots, p-1$, then

$$l^r k - k l^r = \sum_{j=0}^{r-1} l^j (lk - kl) l^{r-1-j}$$
$$= \sum_{j=0}^{r-1} l^j \nu l^{r-1-j}(z-1).$$

Therefore

$$(l^r k - k l^r)(z-1)^{p-2} = r l^{r-1} \nu (z-1)^{p-1}. \tag{2.3}$$

3. The main result. Let K be a field with odd characteristic p. Let G, $l = l(\alpha)$, k, ν be as in the previous section. Let W be the left ideal in KG given as $W = KGl + KG(z-1)$. We define $M = M(\alpha) = KG/W$. Then M is a cyclic KG-module generated by $m = 1 + W$ where $(z-1)m = 0$ and $(y-1)m = \alpha(x-1)m$. The dimension of M is p, and the restriction of M to a $K\langle x\rangle$-module is isomorphic to $K\langle x\rangle$.

THEOREM 3.1. *If $\alpha \notin F_p$ (the prime field), then $M(\alpha)$ is a periodic KG-module with period $2p$.*

The proof consists of constructing a minimal free resolution for M. Let $F = KGa \oplus KGb$ be a free module with generators a and b. For each $i = 1, \ldots, p-1$, let

$$m(i, 1) = l^i a - (z-1)b,$$
$$m(i, 2) = k(z-1)^{p-2} a - l^{p-i} b.$$

Define M_i to be the submodule of F generated by $m(i, 1)$ and $m(i, 2)$.

LEMMA 3.2. Dim $M_i \geqslant p^3 + p$.

PROOF. Now $\tilde{H}m(i, 1) = (y-1)^i \tilde{H}a$, and $\tilde{H}m(i, 2) = -(y-1)^{p-i}\tilde{H}b$. Therefore the KH-module

$$E = \sum_{j=0}^{p-i-1} KH(y-1)^j m(i, 1) \oplus \sum_{j=0}^{i-1} KH(y-1)^j m(i, 2)$$

is a free KH-submodule of $(M_i)_H$ whose KH-socle is the subspace with a basis consisting of the elements $(y-1)^t \tilde{H}a$, $t = i, \ldots, p-1$, and $(y-1)^s \tilde{H}b$, $s = p-i, \ldots, p-1$. Also Dim $E = p^3$. Let

$$m(i, 3) = k(z-1)^{p-2} m(i, 1) - l^i m(i, 2)$$
$$= (kl^i - l^i k)(z-1)^{p-2} a$$
$$= -i l^{i-1} \nu (z-1)^{p-1} a.$$

The last equality follows from (2.3). By Lemma 2.2,

$$(x-1)^{p-1}\nu^{-1}m(i,3) = -i(y-1)^{i-1}\tilde{H}a \notin E.$$

Let

$$L = KH\,\nu^{-1}m(i,3) = K\langle x\rangle\nu^{-1}m(i,3).$$

Then $L \cap E = 0$ and $\operatorname{Dim} L = p$. Consequently $L \oplus E$ is a subspace of M_i of dimension $p^3 + p$.

LEMMA 3.3. *$M_1 \cong \Omega^2(M)$ and* $\operatorname{Dim} M_1 = p^3 + p$.

PROOF. From the definition we know that $W \cong \Omega(M)$. We have an exact sequence

$$0 \to \Omega^2(M) \to F \xrightarrow{\varphi} W \to 0$$

where φ is defined by $\varphi(a) = z - 1$ and $\varphi(b) = l$. Then

$$\varphi(m(1,1)) = l(z-1) - (z-1)l = 0.$$

Also $\varphi(m(1,2)) = k(z-1)^{p-1} - l^p = 0$, by Lemma 2.1. Hence M_1 is in the kernel of φ. Since $\operatorname{Dim} W = p^3 - p$, the dimension of the kernel of φ is $p^3 + p$. By Lemma 3.2, M_1 is the kernel of φ.

LEMMA 3.4. *For each $i = 1, \ldots, p-2$, $\Omega^2(M_i) \cong M_{i+1}$. Moreover* $\operatorname{Dim} M_i = p^3 + p$ *for all $i = 1, \ldots, p-1$.*

PROOF. Assume, by induction, that $\operatorname{Dim} M_i = p^3 + p$. Note that

$$l^{p-i}m(i,1) - (z-1)m(i,2) = 0. \tag{3.5}$$

We also have that

$$\begin{aligned} lk(z-1)^{p-2}m(i,1) - l^{i+1}m(i,2) &= -l(l^ik - kl^i)(z-1)^{p-2}a \\ &= -i\nu(z-1)^{p-1}m(i,1). \end{aligned}$$

Therefore

$$[lk + i\nu(z-1)](z-1)^{p-2}m(i,1) - l^{i+1}m(i,2) = 0. \tag{3.6}$$

Let $F' = KGc \oplus KGd$ be the free KG-module with generators c and d. We can form the exact sequence

$$0 \to \Omega(M_i) \to F' \xrightarrow{\psi} M_i \to 0,$$

where $\psi(c) = m(i,1)$ and $\psi(d) = m(i,2)$. By (3.5) and (3.6), the kernel of ψ contains the elements

$$u_1 = l^{p-i}c - (z-1)d$$

and

$$u_2 = [lk + i\nu(z-1)](z-1)^{p-2}c - l^{i+1}d.$$

Now $\tilde{H}u_1 = (y-1)^{p-i}\tilde{H}c$, $\tilde{H}u_2 = (y-1)^{i+1}\tilde{H}d$. Let

$$u_3 = k(z-1)^{p-1}c - l^i(z-1)d = l^iu_1.$$

Clearly $(z-1)^{p-1}u_3 = 0$ and

$$(x-1)^{p-1}(z-1)^{p-2}u_3 = -(y-1)^i\tilde{H}d.$$

By an argument similar to that in Lemma 3.2, we get that the dimension of the module L, generated by u_1 and u_2, is at least $p^3 - p$. Since the dimension of the kernel of ψ is $p^3 - p$, $L \cong \Omega(M_i)$.

We can form the exact sequence

$$0 \to \Omega^2(M_i) \to F \xrightarrow{\theta} L \to 0$$

where $\theta(a) = u_1$ and $\theta(b) = u_2$. It is easy to see that

$$\theta(m(i+1, 1)) = l^{i+1}u_1 - (z-1)u_2 = 0.$$

Also

$$\begin{aligned}\theta(m(i+1, 2)) &= k(z-1)^{p-2}u_1 - l^{p-i-1}u_2 \\ &= \left[kl^{p-i} - l^{p-i}k - i\nu l^{p-i-1}(z-1)\right](z-1)^{p-2}c = 0,\end{aligned}$$

by (2.3). Consequently M_{i+1} is in the kernel of θ. By Lemma 3.2, M_{i+1} is the kernel of θ.

To conclude the proof of Theorem 3.1 we need only the following.

LEMMA 3.7. $\Omega^2(M_{p-1}) \cong M$.

PROOF. We have an exact sequence

$$0 \to \Omega(M_{p-1}) \to F' \xrightarrow{\sigma} M_{p-1} \to 0$$

where $F' = KGc \oplus KGd$, $\sigma(c) = m(p-1, 1)$ and $\sigma(d) = m(p-1, 2)$. Let $u = lc - (z-1)d$. Then $\sigma(u) = 0$. Now $\tilde{H}u = (y-1)\tilde{H}c$ and

$$(x-1)^{p-1}(z-1)^{p-2}l^{p-1}u = -(y-1)^{p-1}\tilde{H}d \neq 0.$$

By an argument similar to that of Lemma 3.2, we get that Dim $KGu \geqslant p^3 - p$. Therefore the kernel of σ is $KGu \cong \Omega(M_{p-1})$.

Define τ: $KG \to KGu$ by $\tau(1) = u$. The kernel of τ has dimension p and is isomorphic to $\Omega^2(M_{p-1})$. Let $\omega = (z-1)^{p-1}l^{p-1}$. Then $\tau(\omega) = 0$ and $(z-1)\omega = l\omega = 0$. Since $(x-1)^{p-1}\omega = (y-1)^{p-1}\tilde{H} \neq 0$, $KG\omega$ is the kernel of τ, and $M = KG\omega$. This completes the proof of the lemma and the theorem.

It should be noted that if $\alpha \in F_p$ then $M(\alpha)$ is not periodic. This follows from the fact that the restriction of $M(\alpha)$ to the subgroup $J = \langle x^{-\alpha}y, z\rangle$ is not a periodic module (see [2]). It remains to show that there exist periodic modules with period $2p$ when $K = F_p$.

Let $f = T^n + \beta_{n-1}T^{n-1} + \cdots + \beta_1 T + \beta_0$ be an irreducible polynomial in $K[T]$. Let L be the KG-module of dimension np on which x and y are represented by the matrices

$$A_x = \begin{bmatrix} I & & & & \\ I & I & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & I & I \end{bmatrix}, \quad A_y = \begin{bmatrix} I & & & & \\ U & I & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & U & I \end{bmatrix},$$

respectively, where I is the $n \times n$ identity matrix, and

$$U = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -\beta_0 & -\beta_1 & \cdot & \cdot & \cdot & -\beta_{n-1} \end{bmatrix}$$

is the companion matrix for f. Now L is an indecomposable KG-module. If K' is an extension of K which splits f, then

$$K' \otimes_K L \cong M(\alpha_1) \oplus \cdots \oplus M(\alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are the roots of f in K'. If therefore $n > 1$, then L is periodic with period $2p$ since, by the Noether-Deuring Theorem (see [3, 29.7]),

$$K' \otimes \Omega^{2p}(L) \cong \Omega^{2p}(K' \otimes L) \cong K' \otimes L$$

implies that $\Omega^{2p}(L) \cong L$.

The reader is invited to check that L is periodic with period $2p$ when the matrix U is replaced by

$$U' = \begin{bmatrix} \alpha & & & & \\ 1 & \alpha & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \alpha \end{bmatrix}$$

for $\alpha \in K$, $\alpha \notin F_p$. Combining this with the fact that $M(\alpha) \cong M(\beta)$ if and only if $\alpha = \beta$, we get the following.

THEOREM 3.8. *Let p be an odd prime and let K be a field of characteristic p. If G is the nonabelian group of order p^3 and exponent p, then there exist periodic KG-modules which have period $2p$. Moreover there exist an infinite number of isomorphism classes of such modules and there exist such modules with arbitrarily large dimension.*

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