THE STRONGLY PRIME RADICAL

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Abstract. Let $R$ denote a strongly prime ring. An explicit construction is given of the radical in $R$-mod corresponding to the unique maximal proper torsion theory. This radical is characterized in two other ways analogous to known descriptions of the prime radical in rings. If $R$ is a left Ore domain the radical of a module coincides with the torsion submodule.

1. The strongly prime radical. The terminology of radicals in modules is that of Stenstrom [3]. Throughout this paper all rings have a unity and all modules are unital left modules. For a ring $R$ the category of $R$-modules is denoted by $R$-mod.

A functor $\sigma: R$-mod $\rightarrow R$-mod is called a preradical if $\sigma(M)$ is a submodule of $M$ and $\sigma(M)\alpha \subseteq \sigma(N)$ for each morphism $M \rightarrow N$ in $R$-mod. A preradical $\alpha$ is called a radical if $\sigma(M/\alpha(M)) = 0$ for all $M \in R$-mod. A preradical $\sigma$ is called left exact if $\sigma(N) = N \cap \sigma(M)$ whenever $N \subseteq M$ in $R$-mod (equivalently, if $\sigma$ is a left exact functor). One method of constructing left exact radicals is given by the following result.

Proposition 1. Let $\mathcal{M}$ be any nonempty class of modules closed under isomorphisms. For any module $M$ define

$$\sigma(M) = \bigcap \{K | K \subseteq M, M/K \in \mathcal{M}\}.$$  

It is assumed that $\sigma(M) = M$ if $M/K \notin \mathcal{M}$ for all $K \subseteq M$. Then

1. $\sigma[M/\sigma(M)] = 0$ for all modules $M$;
2. if $\mathcal{M}$ is closed under taking nonzero submodules, $\sigma$ is a radical;
3. if $\mathcal{M}$ is closed under taking essential extensions, then $\sigma(M) \cap N \subseteq \sigma(N)$ for all submodules $N \subseteq M$.

In particular, $\sigma$ is a left exact radical if $\mathcal{M}$ is closed under nonzero submodules and essential extensions.

Proof. The proofs of (1) and (2) are straightforward and so are omitted; the last sentence follows from (2) and (3). To prove (3) let $N \subseteq M$ be modules. We must verify that $N \cap \sigma(M) \subseteq K$ whenever $N/K \in \mathcal{M}$. By Zorn's lemma, choose $W$ maximal in

$$\mathcal{S} = \{W | K \subseteq W \subseteq M, W \cap N = K\}.$$
We claim that $(W + N)/W$ is essential in $M/W$. For if

$$X/W \cap (W + N)/W = 0$$

where $X/W \neq 0$ then $X \subseteq W$ so $X \cap N \subseteq K$ by the choice of $W$. Suppose $x \in (X \cap N) - K$. Then

$$x + W \in X/W \cap (W + N)/W = 0$$

so $x \in N \cap W = K$, a contradiction. Hence $(W + N)/W \subseteq M/W$ is essential. Since

$$(W + N)/W \cong N/(W \cap N) = N/K \in \mathcal{R},$$

it follows that $M/W \in \mathcal{R}$ and so $\sigma(M) \subseteq W$. Thus $\sigma(M) \cap N \subseteq W \cap N = K$ as required. □

We are going to apply this to the following class of modules: An $R$-module $M$ is called strongly prime [1] if $M \neq 0$ and, for each nonzero element $m \in M$, there exists a finite subset $(r_1, \ldots, r_k) \subseteq R$ (depending on $m$) such that $r_i m = 0$ for all $i$ ($r \in R$) implies $r = 0$. In [1] the set $(r_1, r_2, \ldots, r_k)$ is called an insulator for $m$. A ring $R$ is called left strongly prime if $_R R$ is strongly prime (this is not left-right symmetric [1, p. 212]).

**Proposition 2.** The class of strongly prime modules is closed under taking isomorphic images, (nonzero) submodules and essential extensions.

**Proof.** It is obviously closed under isomorphic images and nonzero submodules. If $M \subseteq X$ is an essential extension and $M$ is strongly prime let $0 \neq x \in X$. Then $Rx \cap M \neq 0$, say $0 \neq rx \in M$, $r \in R$. Then if $(r_1, \ldots, r_k)$ is an insulator for $rx$ it is clear that $(r_1 r, \ldots, r_k r)$ is an insulator for $x$. □

Now define the strongly prime radical $\beta$ on $R$-mod by

$$\beta(M) = \cap \{K | K \subseteq M, M/K \text{ strongly prime},$$

where we assume that $\beta(M) = M$ whenever $M$ has no strongly prime images. Observe that every strongly prime module is faithful. If $M$ is strongly prime and $0 \neq r \in R$ then $rM \neq 0$, say $rm \neq 0, m \in M$. If $(r_1, r_2, \ldots, r_k)$ is an insulator for $rm$ then it is also an insulator for $r$ in $_R R$. It follows that $R$ is left strongly prime if and only if it has a strongly prime module [1, p. 220]. In particular, $\beta(M) = M$ for all $M \in R$-mod unless $R$ is a strongly prime ring. In this case Propositions 1 and 2 give:

**Proposition 3.** If $R$ is left strongly prime then $\beta$ is a left exact preradical on $R$-mod.

If $\sigma$ and $\rho$ are two preradicals on $R$-mod, we say that $\rho$ is larger than $\sigma$ (written $\rho > \sigma$) if $\rho(M) \supseteq \sigma(M)$ for every module $M \in R$-mod. Then we have:

**Theorem 1.** Let $R$ be left strongly prime. Then $\beta(R) = 0$ and $\beta > \sigma$ for every left exact preradical $\sigma$ on $R$-mod such that $\sigma(R) = 0$.  

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The strongly prime radical

**Proof.** Clearly $\beta(R) = 0$ (since $R$ is strongly prime). Suppose $\sigma$ is a preradical on $R$-mod for which $\sigma(R) = 0$. Given $M \in R$-mod we must show $\sigma(M) \subseteq \beta(M)$. If not then $\sigma(M) \subseteq K$ for some $K \subseteq M$ with $M/K$ strongly prime. If $\alpha: M \to M/K$ is the natural map then

$$0 \neq [\sigma(M) + K]/K = \sigma(M)\alpha \subseteq \sigma(M/K)$$

so it suffices to show that $\sigma(M) = 0$ whenever $M$ is strongly prime. Suppose on the contrary that $0 \neq m \in \sigma(M)$. Let $(r_1, \ldots, r_k)$ be an insulator for $m$ and define $\lambda: R \to M^k$ by $r\lambda = (rr_1m, \ldots, rr_km)$. This is an $R$-monomorphism and $\lambda(M) \subseteq \sigma(M)^k \subseteq \sigma(M^k)$. But $\sigma$ is left exact so $\sigma(R) = 0$ implies

$$0 = \sigma(\lambda) = \lambda \cap \sigma(M^k) = \lambda,$$

a contradiction. □

A nonempty class $\mathcal{T}$ of modules is called a pretorsion class if it is closed under quotients and direct sums; if in addition $\mathcal{T}$ has the property that $M/K$, $K \in \mathcal{T}$ imply $M \in \mathcal{T}$, then $\mathcal{T}$ is called a torsion class. A pretorsion class is called hereditary if it is closed under taking submodules. If a preradical $\sigma$ on $R$-mod is given, the class $\mathcal{T}_\sigma = \{M | \sigma(M) = M\}$ is known to be a pretorsion class and the assignment $\sigma \leftrightarrow \mathcal{T}_\sigma$ is a bijection between left exact preradicals and hereditary pretorsion classes [3, p. 138] under which left exact radicals correspond with hereditary torsion classes [3, p. 139]. In particular, if $R$ is left strongly prime and $\beta$ is the strongly prime radical on $R$-mod, then Theorem 1 implies that $\mathcal{T}_\beta$ is the torsion class of the largest hereditary torsion theory [3, p. 141] on $R$-mod for which $R$ is torsion-free. The existence of a unique maximal proper torsion theory on $R$-mod was given in [1, p. 220].

2. Further characterizations of the strongly prime radical. In this section we present two characterizations of the strongly prime radical which are analogs of well-known descriptions of the prime radical of a ring. The first gives a generalization of the notion of an $m$-system. A subset $X$ of an $R$-module $M$ is called an $fm$-system if $X \neq \emptyset$ and for each $x \in X$ there is a finite subset $F \subseteq R$ (depending on $x$) such that $rFx \cap X \neq \emptyset$ for all $0 \neq r \in R$.

**Proposition 4.** If $N \subseteq M$ are modules then $M/N$ is strongly prime if and only if $M - N$ is an $fm$-system.

**Proof.** If $M - N$ is an $fm$-system then $M/N \neq 0$ and, if $m \in N$ for some $m \in M$, choose $F = \{r_1, \ldots, r_k\} \subseteq R$ such that $rFm \cap (M - N) \neq \emptyset$ for all $0 \neq r \in R$. Then $F$ is an insulator for $m + N$. For the converse, reverse the argument. □

One immediate consequence of this proposition is that subdirect products of strongly prime modules are strongly prime. Alternatively, if $K_i \subseteq M$, $i \in I$, are submodules such that $M/K_i$ is strongly prime for each $i \in I$, then $M/ \cap K_i$ is strongly prime. This follows since $M - \cap K_i = \cup (M - K_i)$ and the union of a collection of $fm$-systems is again an $fm$-system. In
particular, either $\beta(M) = M$ or $M/\beta(M)$ is strongly prime for every module $M$.

**Theorem 2.** Let $R$ be a left strongly prime ring and let $\beta$ denote the strongly prime radical in $R$-mod. Then

$$\beta(M) = \{ m \in M \mid \text{each fm-system} \ X \text{ with } m \in X \text{ has } 0 \in X \}$$

holds for each module $M$. Furthermore $\beta(M)$ is the unique smallest submodule of $M$ with the property that $M/\beta(M)$ is strongly prime or zero.

**Proof.** The last sentence follows by the preceding remark. Write

$$B = \{ m \in M \mid \text{each fm-system} \ X \text{ with } m \in X \text{ has } 0 \in X \}.$$ 

If $\beta(M) \neq M$ then $M - \beta(m)$ is an fm-system which does not contain zero so $B \subset \beta(M)$ in this case. This clearly holds if $\beta(M) = M$.

Now suppose $m \notin B$; we must show $m \notin \beta(M)$. There is an fm-system $X$ with $m \in X$ and $0 \notin X$. Let $S = \{ K \subseteq M \mid K \text{ a submodule and } K \cap X = \emptyset \}$. Then $0 \in S$ and, by Zorn’s lemma, we may choose a maximal member $K$ of $S$. Since $m \notin K$ we are finished if we can show that $M/K$ is strongly prime, equivalently that $M - K$ is an fm-system. Given $m_1 \in M - K$ then $rm_1 + K$ meets $X$ by the maximality of $K$, say $rm_1 + k = x \in X$. Since $X$ is an fm-system, choose a finite set $F = \{ r_1, \ldots, r_l \} \subseteq R$ such that $sFx \cap X \neq \emptyset$ for each $0 \neq s \in R$. If $sr_1 x \in X$ for such an $s$, then $sr_1 rm_1 + sr_1 k \in X$. But $K \cap X = \emptyset$ and $sr_1 k \in K$ so it follows that $sr_1 rm_1 \notin K$. Thus

$$sr_1 rm_1 \in s(Fr)m_1 \cap (M - K)$$

and so $M - K$ is an fm-system as required. □

Note that this argument yields slightly more. If $X_0$ is any fm-system with $0 \notin X_0$ then Zorn’s lemma produces a maximal fm-system $X \supseteq X_0$ with $0 \notin X$. If we now choose $K$ as in the proof of Theorem 2 then $X \subseteq M - K$ (since $X \cap K = \emptyset$) and hence $X = M - K$ by the maximality of $X$. Thus

**Corollary.** If $X$ is a maximal fm-system such that $0 \notin X$ in a module $M$ then $K = M - X$ is a submodule with $M/K$ strongly prime. In particular, a module $M$ contains an fm-system $X$ with $0 \notin X$ if and only if $M$ has a strongly prime image.

We now turn to a characterization of the strongly prime radical which is analogous to the lower radical construction of the prime radical of a ring. Given a module $M$, inductively define an ascending chain of submodules $M_\lambda$, $\lambda$ an ordinal, as follows:

1. $M_0 = 0$;
2. if $\lambda$ is a limit ordinal, define $M_\lambda = \bigcup_{\mu < \lambda} M_\mu$;
3. if $\lambda = \mu + 1$, define

$$M_\lambda = M_{\mu + 1} = \left\{ m \in M \mid \text{given a finite nonempty subset} \ F \subseteq R, \text{there exists } 0 \neq r \in R \text{ such that } rFm \subseteq M_\mu \right\}.$$
Clearly $M_\lambda \subseteq M_{\lambda+1}$ so these $M_\lambda$ are an ascending chain of submodules. If $\gamma$ is the least ordinal for which $M_\gamma = M_{\gamma+1}$ write $M_\gamma = L(M)$.

**Theorem 3.** If $R$ is left strongly prime and $\beta$ is the strongly prime radical in $R$-mod then $\beta(M) = L(M)$ holds for every $M \in R$-mod.

**Proof.** Let $L(M) = M_\gamma = M_{\gamma+1}$. We show first that $\beta(M) \subseteq M_\gamma$. If $M_\gamma = M$ this is clear. Otherwise it suffices to show $M/M_\gamma$ is strongly prime. If $m \in M - M_\gamma$ then $m \not\in M_{\gamma+1}$ so there exists a finite set $F = \{r_1, \ldots, r_k\} \subseteq R$ such that $rFm \subseteq M$ for every $0 \neq r \in R$. Thus $rFm \cap (M - M_\gamma) \neq \emptyset$ for all $0 \neq r \in R$ so $M - M_\gamma$ is an $fm$-system as required.

To prove $M_\gamma \subseteq \beta(M)$ we prove inductively that $M_\lambda \subseteq \beta(M)$ holds for every ordinal $\lambda$. The only case where proof is required is when $\lambda = \mu + 1$ for some ordinal $\mu$. Assume $M_\mu \subseteq \beta(M)$ and suppose $m \in M_\lambda - \beta(M)$. Then, since $M - \beta(M)$ is an $fm$-system, there exists a finite set $F \subseteq R$ with $rFm \cap (M - \beta(M)) = \emptyset$ for all $0 \neq r \in R$. But $m \in M_\lambda = M_{\mu+1}$ means there exists $0 \neq r_0 \in R$ such that $r_0Fm \subseteq M_\mu$. This contradiction shows that $M_\lambda \subseteq \beta(M)$ and so completes the induction. $\square$

One important class of strongly prime rings is the class of domains. We now relate $\beta(M)$ for $M \in R$-mod to the set of torsion elements $\tau(M)$ where $R$ is a domain. Recall that $\tau(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$.

**Proposition 5.** If $R$ is a domain then $\beta(M) \subseteq \tau(M)$ for all $M \in R$-mod.

**Proof.** We use Theorem 3 and show inductively that $M_\lambda \subseteq \tau(M)$ for every ordinal $\lambda$. Again we need only discuss the case when $\lambda = \mu + 1$ for some ordinal $\mu$. Assume $M_\mu \subseteq \tau(M)$ and suppose $m \in M_\lambda - \beta(M)$. Then, since $M - \beta(M)$ is an $fm$-system, there exists a finite set $F \subseteq R$ with $rFm \cap (M - (M - \beta(M))) = \emptyset$ for all $0 \neq r \in R$. But $m \in M_\lambda = M_{\mu+1}$ means there exists $0 \neq r_0 \in R$ such that $r_0Fm \subseteq M_\mu$. This contradiction shows that $M_\lambda \subseteq \beta(M)$ and so completes the induction. $\square$

In the case of left Ore domains, Levy [2] has shown that, for each $M \in R$-mod, $\tau(M)$ is a submodule of $M$. In this case it is easy to verify that $\tau$ is a left exact radical on $R$-mod and it is clear that $\tau(R) = 0$. Hence, by Theorem 1, $\beta \geq \tau$. With Proposition 5 this gives:

**Proposition 6.** If $R$ is a left Ore domain, then $\tau(M) = \beta(M)$ for all $M \in R$-mod, that is $\tau = \beta$ on $R$-mod.

3. **The faithful prime radical.** The preceding work can be repeated to deal with the radical determined by the class $\mathfrak{P}_0$ of faithful prime modules in $R$-mod (so we assume $R$ is a prime ring). Then Proposition 2 is valid for $\mathfrak{P}_0$ and yields a left exact radical

$$\beta_0(M) = \{K \mid K \subseteq M, M/K \text{ is faithful and prime}\}$$

when we set $\beta_0(M) = M$ if $M$ has no faithful, prime images. We call $\beta_0(M)$ the faithful prime radical of $M$. Clearly $\beta_0 \leq \beta$ over a strongly prime ring.

Define an $m$-system in a module $M$ to be a nonempty subset $X$ of $M$ such that, for each $x \in X$ and $0 \neq r \in R$, $rRx \cap X \neq \emptyset$. Then $M/N$ is a faithful
prime module if and only if \( M - N \) is an \( m \)-system. Furthermore Theorem 2 has its analog for prime rings \( R \) obtained by replacing \( \beta \), "strongly prime" and "\( fm \)-system" by \( \beta_0 \), "faithful prime" and "\( m \)-system" throughout. The proof is analogous to the above and is omitted.

We also have a lower radical construction of \( \beta_0(M) \). A sequence \( M_\lambda, \lambda \) an ordinal, of submodules of a module \( M \) is defined as before except that, when \( \lambda = \mu + 1 \), we define \( M_{\mu+1} = \{ m \in M \mid \text{there is an ideal } I \neq 0 \text{ of } R \text{ with } Im \subseteq M_\mu \} \). Again we find that the terminal module in this ascending chain is \( \beta_0(M) \).

Finally, let \( F \) be a field with a monomorphism \( \alpha: F \to F \) which is not onto and let \( R = F[x, \alpha] \) be the skew polynomial ring with coefficients written on the left. Then \( R \) is a left Ore domain which is left primitive. In fact, if \( b \in F - Fa \), then \( M = R/R(x + b) \) is a faithful irreducible module which is torsion. Hence \( \beta_0(M) = 0 \) while \( \tau(M) = \beta(M) = M \) and so \( \beta_0 < \beta \).

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