DYNAMICAL SYSTEMS WHOSE ORBIT SPACES ARE NEARLY HAUSDORFF

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ABSTRACT. Consider a continuous flow on a locally compact, separable, metric space. If the set of nonperiodic recurrent points is nowhere dense, then there is an open, dense, invariant subset of the phase space which has a Hausdorff orbit space. A separatrix is defined to be a trajectory which is in the closure of the set of trajectories at which the orbit space is not Hausdorff. If the flow is completely unstable, then the set of points which lie on separatrices is nowhere dense in the phase space.

Introduction. Orbit spaces and their topological properties arise in a natural way when the trajectorial structure of a continuous flow is studied, e.g., in [2], [4], and [6]. One of the fundamental concepts of trajectorial structure is that of a parallelizable flow. Parallelizable flows have been studied and characterized in [1], [2], and [7, §2.4]. If the flow is parallelizable, then the natural mapping of the phase space onto the orbit space is the projection of a fiber bundle. One immediate consequence of parallelizability is that the orbit space is Hausdorff. If a flow on a locally compact, separable, metric space has a Hausdorff orbit space, then there is a natural generalization of the fiber bundle structure for parallelizable flows [6]. Hence, flows with Hausdorff orbit spaces are a natural generalization of flows which are parallelizable. Flows with Hausdorff orbit spaces also arise in the study of completely unstable flows [4]. Clearly, it is very restive to assume that a flow has a Hausdorff orbit space. In this paper we will show that if a flow on a locally compact, separable, metric space has a nowhere dense set of nonperiodic recurrent points, then there is an open, dense, invariant subset U of the phase space such that the orbit space with respect to U is Hausdorff. A separatrix is a trajectory which is in the closure of the set of trajectories at which the orbit space is not Hausdorff. In the case that the flow is completely unstable, we will show that the set of points which lie on separatrices is nowhere dense in the phase-space.

1. Basic definitions and properties of continuous flow. Throughout this paper X will be a locally compact, separable, metric space and R, R^+ will denote the reals and nonnegative reals, respectively.

A *continuous flow* π on X is a continuous mapping of $X \times R$ onto X

satisfying the following axioms (where $x\pi t = \pi(x, t)$ for $(x, t) \in X \times R$):

(1) $x\pi 0 = x$ for $x \in X$,

(2) $(x\pi t)\pi s = x\pi(t + s)$ for $x \in X$ and $t, x \in R$.

For $A \subset X$ and $B \subset R$, $A\pi B$ will denote the set $\{x\pi t: x \in A, t \in B\}$. In the special case $B = R$ we will write $C(A)$ instead of $A\pi R$. If A consists of a single element x, we will write $x\pi B$ instead of $\{x\}\pi B$. For $x \in X$, $C(x)$ is called the trajectory through x. A set $A \subset X$ is said to be *invariant* if $C(A) = A$. If $C(x) = \{x\}$, x is called a *critical point*. The set of all critical points will be denoted by S. If $x \in X$ is noncritical and $x\pi t = x$ for some $t \neq 0$, then x is said to be *periodic*. The set of all periodic points will be denoted by $\mathcal{P}$. If $x \in \mathcal{P}$ and X is Hausdorff, then $t_x = \inf\{t > 0: x\pi t = x\}$ exists and is positive. The mapping defined by $x \to t_x$ is called the *fundamental period function*.

If $x, y \in X$, we write xCy if and only if $x \in C(y)$. Evidently C is an equivalence relation on X. The topology of X/C will be the quotient topology. X/C is called the *orbit space* of π. If $A \subset X$ is invariant we will write A/C for the orbit space of $\pi|_{(A \times R)}$.

Let $x \in X$. Then $L^+(x)$ $[L^-(x)]$ will denote the *positive* [negative] *limit set* of x. A flow is said to be *unstable* if no limit set is compact. An $x \in X$ is called *recurrent* if $\overline{C(x)} = L^+(x) = L^-(x)$. $\mathcal{R}$ will denote the set of all recurrent points. An $x \in X$ is called *wandering* if there is a neighborhood U of x and an $s \in R$ such that $(U\pi t) \cap U = \varnothing$ for all $t > s$. W will denote the set of all wandering points. Evidently W is open. A flow is called *completely unstable* if every $x \in X$ is wandering, i.e., $W = X$.

The *prolongation* $D(x)$ of x is $D(x) = D^+(x) \cup D^-(x)$, where $D^+(x)$ is the set $\{y \in X$: there exist sequences $\{x_i\}$ in X and $\{t_i\}$ in R^+ such that $x_i \to x$ and $x_i\pi t_i \to y\}$ and $D^-(x)$ is defined analogously. It can be shown that X/C is Hausdorff if and only if $C(x) = D(x)$ for every $x \in X$ (e.g., see [6, Proposition 2.4]).

The *prolongational limit set* $J(x)$ of x is $J(x) = J^+(x) \cup J^-(x)$ where $J^+(x)$ is the set $\{y \in X$: there exist sequences $\{x_i\}$ in X and $\{t_i\}$ in R^+ such that $x_i \to x$, $t_i \to \infty$, and $x_i\pi t_i \to y\}$ and $J^-(y)$ is defined analogously. π is said to be *parallelizable* if there exist a subset K of X and a homeomorphism $h: X \to K \times R$ such that $K\pi R = X$ and $h(x\pi t) = (x, t)$ for every $(x, t) \in K \times R$. If X is a locally compact, separable, metric space, then π is parallelizable if and only if $J(x) = \varnothing$ for each $x \in X$ [1, Theorem 3].

A subset T of X is called a *local section* of extent λ, $\lambda > 0$, if $(T\pi t) \cap T = \varnothing$ for $0 < |t| < \lambda$. T is said to *generate neighborhoods* of $x \in T$ if for arbitrarily small $a > 0$, $T\pi(-a, a)$ is a neighborhood of x. T is said to be *interior* in X if, for sufficiently small $a > 0$, $T\pi(-a, a)$ is an open subset of X. If $x \in X$ is a noncritical point of π and X is a Tychonoff space, then there exists a local section T which generates neighborhoods of x [2, §VI, Theorem 2.12]. It can be shown that T may be chosen to be interior in X. If T is an

interior local section and U an open set, then $T \cap U$ is an interior local
section.

If U is a subset of X, then int U and ∂U will denote the interior and the
boundary of U respectively.

2. Orbit spaces.

PROPOSITION 2.1. *If $W \neq \varnothing$, then there is an open, dense, invariant subset
W_1 of W such that $J(W_1) \cap W_1 = \varnothing$.*

PROOF. Evidently $x \in W$ if and only if there is an invariant neighborhood
U of x such that $J(U) \cap U = \varnothing$. Set $F = \{U \subset W: U$ open and invariant,
$J(U) \cap U = \varnothing\}$. Let $\{U_i\}$ be a chain in F and set $V = \bigcup U_i$. Since $J(U_i) \cap
U_i = \varnothing$ for each i, it is easy to verify that $J(V) \cap V = \varnothing$. Hence, V is a
maximal element for $\{U_i\}$. By Zorn's lemma there is a maximal element W_1
of F. Suppose W_1 is not dense in W. Then there exist an $x \in W - W_1$ and
an open invariant neighborhood $U_0 \subset W - \overline{W}_1$ of x such that $J(U_0) \cap U_0 =
\varnothing$. Then $U_0 \cup W_1 \in F$. This contradicts the maximality of W_1. Hence, W_1 is
dense in W.

PROPOSITION 2.2. *If* int $\mathcal{P} \neq \varnothing$, *then the set $\mathcal{P}_1$ of points at which the
fundamental period function is continuous is an open, dense subset of* int $\mathcal{P}$.

PROOF. Let $\Upsilon(x)$ denote the fundamental period of $x \in \mathcal{P}$. It is known and
easy to prove that $\lim \inf_{y \to x} \Upsilon(y) \geqslant \Upsilon(x)$, i.e., Υ is lower semicontinuous.
Theorem 1 [3, p. 394] states: "The set D of points of discontinuity of a B
measurable function of class 1 is of the first category". Hence, Υ is continu-
ous on a dense subset of $\mathcal{P}$. Let Υ be continuous at $x \in$ int $\mathcal{P}$ and let T be
an interior local section of extent λ which generates neighborhoods of x and
is such that $\overline{T} \cap C(x) = \{x\}$. We will first show that there is an open
neighborhood U of x such that $C(y) \cap T = \{y\}$ for every $y \in U \cap T$.
Suppose the contrary. Then there exists a sequence $\{x_i\}$ in $\mathcal{P} \cap T$ such that
$x_i \to x$ and $x_i \pi t_i \in T$ for some $t_i \in (0, \Upsilon(x_i))$. Since $y\pi(\Upsilon(y) - a) = y\pi(-a)$
for every $a \in R$ and T has extent λ, eventually we must have

$$t_i \in [\lambda, \Upsilon(x_i) - \lambda] \subset [\lambda, \Upsilon(x) - \tfrac{1}{2}\lambda].$$

Let $\{s_i\}$ be any convergent subsequence of $\{t_i\}$ with $s_i \to s \in [\lambda, \Upsilon(x) - \tfrac{1}{2}\lambda]$.
Then $x\pi s \leftarrow x_i \pi s_i \in T$. This contradicts the assumption that $\overline{T} \cap C(x) =
\{x\}$. Hence, there is a neighborhood U of x such that $C(y) \cap T = \{y\}$ for
every $y \in T \cap U$. Without loss of generality we may assume that T was
chosen originally so that $T \subset U \cap (\text{int } \mathcal{P})$. We will now show that Υ is
continuous at each point of T. Let $y \in T$ and $\{y_i\}$ be a sequence in X such
that $y_i \to y$. Since T is interior and $T \subset$ int $\mathcal{P}$ we may assume that $y_i \in C(T)
\cap \mathcal{P}$ for each i. For each $a \in (0, \tfrac{1}{2}\Upsilon(y))$, $T\pi(-a, a)$ is a neighborhood of
$y = y\pi\Upsilon(y)$. Eventually $T\pi(-a, a)$ is a neighborhood of y_i and $y_i\pi\Upsilon(y)$. Then
there exist $s_i, t_i \in (-a, a)$ such that $y_i\pi s_i \in T$ and $(y_i\pi\Upsilon(y))\pi t_i \in T$. Since

$$(y_i \pi \Upsilon(y)) \pi t_i = (y_i \pi s_i) \pi (\Upsilon(y) + t_i - s_i)$$

and

$$C(y_i \pi s_i) \cap T = \{ y_i \pi s_i \}$$

we must have $\Upsilon(y_i)$ is a positive integer multiple of $\Upsilon(y) + t_i - s_i$. Hence, eventually $\Upsilon(y_i) \leqslant \Upsilon(y) + 2a$. Since $a \in (0, \frac{1}{2}\Upsilon(y))$ was arbitrary we have $\lim \sup_{z \to y} \Upsilon(z) \leqslant \Upsilon(y)$. This coupled with the lower semicontinuity of Υ yields the continuity of Υ on T and, hence, on $C(T)$ which is an open neighborhood of x. The desired result follows.

PROPOSITION 2.3. *Let* W_1 *and* $\mathcal{P}_1$ *be as in Propositions 1 and 2. Set* $\mathbb{S}_1 = \text{int } \mathbb{S}$. *Then each of* W_1, $\mathcal{P}_1$, *and* $\mathbb{S}_1$ *has a Hausdorff orbit space.*

PROOF. Since $J(W_1) \cap W_1 = \varnothing$, the flow $\pi|_{W_1}$ is parrallelizable. It follows directly that W_1/C is Hausdorff. Since Υ is continuous on $\mathcal{P}_1$ it is easy to show that $C(x) = D(x)$ for every $x \in \mathcal{P}_1$. Hence, $\mathcal{P}_1/C$ is Hausdorff. Likewise it is easy to show that $C(x) = D(x)$ for every $x \in \mathbb{S}_1$ so that $\mathbb{S}_1/C$ is Hausdorff.

THEOREM 2.4. *Let* π *be a continuous flow on a locally compact, separable, metric space. Then there is an open, dense, invariant subset* U *of* $X - (\mathcal{R} - (\text{int}\mathcal{P} \cup \text{int}\mathbb{S}))$ *such that the orbit space of* U *is Hausdorff.*

PROOF. In [5, p. 29] it is shown that $X = \overline{W \cup R}$. Let W_1, $\mathcal{P}_1$ and $\mathbb{S}_1$ be as in Propositions 1–3. Then

$$X = \overline{W} \cup \overline{\mathcal{R}} = \overline{W}_1 \cup \overline{\mathcal{P}}_1 \cup \overline{\mathbb{S}}_1 \cup \overline{(\mathcal{R} - (\text{int } \mathcal{P}_1 \cup \text{int } \mathbb{S}_1))} .$$

It follows directly that $U = W_1 \cup \mathcal{P}_1 \cup \mathbb{S}_1$ is an open dense subset of $X - (\mathcal{R} - (\text{int } \mathcal{P} \cup \text{int } \mathbb{S}))$. Since W_1, $\mathcal{P}_1$, $\mathbb{S}_1$ are open, disjoint, invariant sets with Hausdorff orbit spaces, it follows that U has a Hausdorff orbit space.

3. Separatrices.

DEFINITION 3.1. Two trajectories C_1 and C_2 are said to be inseparable if any two neighborhoods of C_1 and C_2 in X/C have nonempty intersection. A separatrix is a trajectory which lies in the closure of the inseparable elements of X/C.

In the case that X is a differentiable manifold and the flow is differentiable and unstable this definition of separatrix coincides with that given by Markus in [4]. Roughly speaking the trajectories at which the orbit space fails to be Hausdorff are separatrices.

THEOREM 3.2. *Let* π *be a completely unstable flow on a locally compact, separable, metric space* X. *The set of separatrices is nowhere dense in* X.

PROOF. Let S and S_1 denote the set of separatrices and the set of inseparable elements of X/C respectively. Then $\overline{S}_1 = S$. Set $F = \{ U \subset X : U$ open and invariant, $U \neq X$, $J(U) \cap U = \varnothing \}$. It is easy to show that if x is a

wandering point, then there exists a neighborhood V of x such that $J(V) \cap V = \emptyset$. Hence, F covers X. Since X is Lindelöf there is a countable family $\{U_i\}$ of elements of F which cover X. Let C_1 be any element of S_1. Then there is a $C_2 \in S_1$ such that C_1 and C_2 are inseparable. It is easy to show that $C_1 \subset J(C_2)$. Let j be such that $C_2 \subset U_j$. Since $J(U_j) \cap U_j = \emptyset$ we must have $C_1 \subset \partial U_j$. It follows that $S_1 \subset \cup_{j=1}^{\infty} \partial U_j$. Since each ∂U_j is nowhere dense, S_1 must be nowhere dense (Baire category theorem). Therefore S is nowhere dense.

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