THE SOLVABILITY OF OPERATOR EQUATIONS WITH
ASYMPTOTIC QUASIBOUNDED NONLINEARITIES

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Abstract. We study the solvability of operator equations involving quasi-bounded and asymptotically quasibounded nonlinear perturbations of linear Fredholm operators.

1. Let $X$ and $Y$ be Banach spaces, $L: X \to Y$ a linear Fredholm map of nonnegative index and $N: X \to Y$ a compact map. The operator equation of the form

$$Tx = Ax + Nx = f$$

has been extensively studied by many authors in recent years. Under various growth conditions on $N$, the surjectivity of $T$ has been proven in a number of papers (see [4], [5], [7] and the references therein).

Alternatively, beginning with a paper of Landesman and Lazer [6], much work has been done on the solvability of equation (1) for a certain range of values of $Pf$, where $P$ is the projection of $Y$ on the cokernel of $A$. Using the stable homotopy arguments, Nirenberg [9], [10], Berger [1], Mawhin [8], Podolak [11], Borisovich, Zvyagin and Sapronov [2] and others have studied equation (1). The alternative method has also been used to study equation (1) (with noncompact $N$ too) in a series of papers by Cesari and his coworkers, Fučík, Kučera and Nečas [5], and many others (cf. the survey paper by Cesari [3] and the monograph by Berger [1] for contributions of other authors). In all these papers (except in [2], [7], [11]) $N$ is assumed to have less than linear or linear growth.

In [2] and [11] the authors have studied equation (1) under the assumption that $N$ is asymptotically linear or asymptotically Lipschitz (i.e., $B$ in Definition 1 below is a Lipschitz map), respectively. In a series of papers Mawhin (cf. [7], [8]) has studied equation (1) with $f \in R(A)$ involving certain quasi-bounded maps $N$ using his coincidence degree.

In this paper we study the surjectivity of $T$ with $N$ either quasibounded or asymptotically quasibounded as defined below. Moreover, in case when the index of $A$, $i(A)$, is zero we provide a new growth condition on $PN|_{\ker A}$ that insures the solvability of equation (1) with these types of nonlinearities $N$. In the proofs of our main results we use a special case of the degree theory for
compact perturbations of nonlinear $C^1$-Fredholm maps as developed in [2] or, equivalently, the stable homotopy arguments since for our map $T$ this degree can be defined in terms of elements of the stable homotopy group $\pi_{n+m}(S^n)$ (see [1], [2], [9]).

2. Set $X_1 = \ker A$ and $Y_2 = A(X)$. Since $A$ is Fredholm, $\dim X_1 = n < \infty$ and $Y_2$ is closed we have the following direct sum decompositions: $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ with $\dim Y_1 = m < \infty$ and $\text{ind}(A) = n - m > 0$. Define a new norm on $X$ by

$$\|x\|_1 = \max\{\|x_1\|, \|x_2\|\},$$

where $x = x_1 + x_2$ with $x_i \in X_i$, $i = 1, 2$. Let $P: Y \to Y_1$ be a linear continuous projection onto $Y_1$, $H$ be the inverse of the linear homeomorphism $A|_{X_2}$; $X_2 \to Y_2$ and $\alpha = \|H\|$.

**Theorem 1.** Suppose that for a given $f$ in $Y$ the following conditions hold:

1. There exist constants $M_f > 0$ and $N_f > 0$ such that $PN(x_1 + x_2) - tf_1 \neq 0$ for $\|x_2\| < r$, $r > N_f$, $\|x_1\| > rM_f$ and $t \in [0, 1]$;
2. $M = H(I - P)N$ is quasibounded, i.e.,

$$|M| = \limsup_{\|x\|_1 \to \infty} \frac{\|Mx\|}{\|x\|_1} < \infty$$

and $|M|\max\{1, M_f\} < 1$;
3. The stable homotopy class $\eta_p$ of $PN|S_p^{n-1}$: $S_p^{n-1} \to Y_1 \setminus \{0\}$, $p > rM_f$, is nontrivial, where $S_p^{n-1} \subset X_1$ is a sphere of radius $p$.

Then equation (1) is solvable for this $f$.

**Proof.** Let $e > 0$ be small. By (2) there exists $R > N_f$ such that

$$\|Mx\| = \|H(I - P)Nx\| < (|M| + \epsilon)\|x\|_1,$$

for all $\|x\|_1 > R$. Moreover, there exists an $r > R$ such that $Ax + t(I - P)Nx - tf_2 \neq 0$ for all $x = x_1 + x_2$ with $\|x_1\| < rM_f$ and $\|x_2\| = r$ and $t \in [0, 1]$. If not, then for each $r > R$ there exist $t \in [0, 1]$ and $x$ with $\|x_1\| < rM_f$ and $\|x_2\| = r$ such that $Ax_2 + t(I - P)Nx - tf_2 = 0$, and therefore

$$\|x_2\| < \|H(I - P)Nx\| + \alpha\|f_2\| < (|M| + \epsilon)\|x\|_1 + \alpha\|f_2\|,$$

or

$$1 < \frac{1}{r}(|M| + \epsilon)\|x\|_1 + \frac{\alpha}{r}\|f_2\| < (|M| + \epsilon)\max\{1, M_f\} + \frac{\alpha}{r}\|f_2\|.$$ 

Passing to the limit as $r \to \infty$, we obtain $1 < (|M| + \epsilon)\max\{1, M_f\}$ which is in contradiction with condition (2) for $\epsilon$ small enough. Hence, an $r$ with the above property exists.

Next, we define $D = \{x = x_1 + x_2 \in X \mid \|x_1\| < rM_f, \|x_2\| < r\}$ with $r$ chosen as above, and define the homotopy $H: [0, 1] \times D \to Y$ by

$$H(t, x) = (Ax + t(I - P)Nx - tf_2, PN(x_1 + tx_2) - tf_1).$$
We claim that \( H(t, x) \neq 0 \) for \( t \in [0, 1] \) and \( x \in \partial D \). Indeed, if \( x \in \partial D \) is such that \( \|x_2\| < r \), then \( \|x_1\| = rM_f \) and by (1), \( PN(x_1 + tx_2) - tf_1 \neq 0 \) for all \( t \in [0, 1] \). If \( x \in \partial D \) is such that \( \|x_1\| < rM_f \), then \( \|x_2\| = r \) and \( Ax + t(I - P)Nx - tf_2 \neq 0 \) for all \( t \in [0, 1] \). Thus, by the homotopy theorem in [2],

\[
\deg(A + N - f, \overline{D}, 0) = \deg(H_0, \overline{D}, 0) = \eta_r,
\]
which, by the solvability property of this degree, implies that \( Ax + Nx = f \) for some \( x \in D \). □

To treat a larger class of nonlinear maps \( N \), we need:

**Definition 1.** A map \( A: X \rightarrow Y \) is said to be **asymptotically quasibounded** if there exists a nonzero continuous quasibounded map \( B: X \rightarrow Y \), i.e.,

\[
|B| = \limsup_{\|x\| \rightarrow \infty} \frac{\|Bx\|}{\|x\|} < \infty
\]

such that

(A) \( \lim_{R \rightarrow \infty} N(Rx)/R = B(x) \) uniformly on bounded sets in \( X \).

Such maps with \( B \) Lipschitz have been studied by Podolak [11].

Theorem 1 admits the following extension:

**Theorem 2.** Suppose that \( N \) satisfies condition (A) and that \( B \) is continuous, satisfies conditions (1) and (3) of Theorem 1 for \( f = 0 \) and that the following condition holds:

(2') \( K = H(I - P)B \) is quasibounded, i.e.,

\[
|K| = \limsup_{\|x\| \rightarrow \infty} \frac{\|Kx\|}{\|x\|} < \infty
\]

and \( |K|\max\{1, M_0\} < 1 \).

Then equation (1) is solvable for each \( f \) in \( Y \).

**Proof.** Since for each \( f \) in \( Y \), \( N_f x = Nx - f \) satisfies condition (A) with the same \( B \), it is sufficient to consider the case \( f = 0 \). Define

\[
\overline{D} = \{x = x_1 + x_2 \in X \mid \|x_1\| < rM_f, \|x_2\| < r\},
\]

where \( r \) is chosen as in Theorem 1 using property (2') of \( K \). For \( R > 0 \), define the map \( H_R: \overline{D} \rightarrow Y \) by

\[
H_R(x) = (1/R)(A(Rx) + (I - P)N(Rx), PN(Rx))
\]

and the homotopy \( H: [0, 1] \times \overline{D} \rightarrow Y \) by

\[
H(t, x) = (Ax + t(I - P)Bx, PB(x_1 + tx_2)).
\]

By our choice of \( r \) we know that \( H(t, x) \neq 0 \) for \( t \in [0, 1] \) and \( x \in \partial D \). Clearly, if \( x \in X \) is a solution of equation (1), then \( u = x/R \in D \) is a solution of \( H_R(u) = 0 \) for \( R \) sufficiently large, and conversely. Moreover, \( \lim_{R \rightarrow \infty} H_R(x) = H(1, x) \) uniformly for \( x \in D \) with \( \|H(1, x)\| > \varepsilon > 0 \) for all \( x \in \partial D \) since \( H(1, \cdot) \) is a proper map. In view of this, it follows that for sufficiently large \( R \), \( H_R(x) \neq 0 \) on \( \partial D \) and

\[
F_R(t, x) = H(1, x) + t(H_R(x) - H(1, x)) \neq 0
\]
for \( t \in [0, 1] \) and \( x \in \partial \mathbb{D} \). The compactness of \( N \) and condition (A) imply that \( B \) is compact and consequently

\[
F_R(t, x) = Ax + (1 - t)Bx + tN(Rx)/R
\]

is an admissible homotopy on \([0, 1] \times \mathbb{D}\) (cf. (4.2) in [2]). Hence,

\[
\deg(H_R, \mathbb{D}, 0) = \deg(H(1, \cdot), \mathbb{D}, 0) = \deg(H(0, \cdot), \mathbb{D}, 0) = \eta_r
\]

which implies that the equation \( H_R(x) = 0 \) is solvable in \( \mathbb{D} \). \( \Box \)

**Remark.** When \( A \) is asymptotically linear, i.e., \( A(x) = B(x) + w(x), x \in X \), for some continuous and linear map \( B: X \to Y \) with \( w(x)/\|x\| \to 0 \) as \( \|x\| \to \infty \), then \( N \) is quasibounded with \( |N| = \|B\| \). Hence, Theorem 1 extends Theorem 4.5 in [2], which is, on the other hand, an abstract extension of some results of Nirenberg [9] involving everywhere bounded nonlinearities \( N \). Other extensions of Nirenberg’s results to sublinear or quasibounded nonlinearities are given in [1, 4], [5], [7], [8] (cf. [1] for other references).

**Remark.** If \( B \) in condition (A) is Lipschitz, i.e., \( \|Bx - By\| < k\|x - y\| \) for all \( x, y \in X \) and some small \( k > 0 \), then condition (1) in Theorem 2 can be replaced by the following easier to verify condition of Podolak [11]:

\[
(1') \quad \|PA(a - x_0)\| > b \text{ for some positive } b \text{ and all } a \in R^n \text{ with } \|a\| = 1,
\]

where \( x_0 = \{x_{01}, \ldots, x_{0n}\} \) is a fixed basis for \( \ker A \) of unit vectors and

\[
a \cdot x_0 = a_1x_{01} + \cdots + a_nx_{0n}.
\]

In this sense Theorem 2 extends Theorem 1 in [11].

Let us now look at a new condition on \( PN|_{X_1} \) which implies that \( \deg(PN|_{X_1}, B(0, r), 0) \neq 0 \) with \( B(0, r) \subset X_1 \). Suppose that \( X \) and \( Y \) are such that there exist a map \( J: X_1 \to Y_1^* \) and a continuous and odd map \( G: X_1 \to Y_1^* \) with \( Gx \neq 0 \) for \( x \neq 0 \) and \( (Gx, Jx) = \|Gx\| \cdot \|Jx\| \) for all \( x \in X_1 \). This is always so if \( Y = X \) or \( Y = X^* \). Indeed, if \( Y_1 = X_1 \), as \( G \) and \( J \) we can take the identity and the normalized duality map, respectively; while, if \( Y_1 = X_1^* \) as \( G \) and \( J \) we can take the normalized duality map and the identity, respectively. The condition in question is:

\[
(4) \quad \|PNx\| + (PNx, Jx)/\|Jx\| > 0 \text{ for } x \in \partial B(0, \rho), \rho > rM_f.
\]

**Corollary 1.** Let \( A \) and \( N \) satisfy conditions (1) and (2) of Theorem 1. Then, if condition (4) holds for all \( \rho > rM_f \) and the index of \( A \) is zero, equation (1) is solvable.

**Proof.** By Theorem 1 it suffices to show that \( \deg(PN, B(0, \rho), 0) \neq 0 \), where \( PN \) is restricted to \( \overline{B}(0, \rho) \). Define the homotopy \( H: [0, 1] \times \overline{B}(0, \rho) \to Y_1 \) by \( H(t, x) = tPNx + (1 - t)Gx \). Then \( H(t, x) \neq 0 \) for \( t \in [0, 1] \) and \( x \in \partial \mathbb{B} \). If not, then \( tPNx + (1 - t)Gx = 0 \) for some \( t \in [0, 1] \) and \( x \in \partial \mathbb{B} \). Since \( t \neq 0,1 \), we have

\[
\|PNx\| + \frac{(PNx, Jx)}{\|Jx\|} = \frac{1 - t}{t} \|Gx\| - \frac{1 - t}{t} \frac{(Gx, Jx)}{\|Jx\|} = 0
\]
in contradiction with condition (4). By the oddness of $G$ we obtain:
\[
\deg(PN, B(0, \rho), 0) = \deg(G, B(0, \rho), 0) \neq 0. \qed
\]

Similarly, using Theorem 2, we obtain:

**Corollary 2.** Let $K$ be asymptotically quasibounded and $B$ satisfy conditions (1) and (2') of Theorem 2 with $f = 0$. Then, if $\text{ind} A = 0$ and $PB$ satisfies condition (4) for $f = 0$, equation (1) is solvable for each $f$ in $Y$.

Under a somewhat stronger condition than (4), we have:

**Theorem 3.** Let $X$ and $Y$ be Banach spaces with $\dim X = \dim Y < \infty$ and let $T: X \to Y$ be continuous and satisfy
\[
(5) \quad \|Tx\| + (Tx, Jx)/\|Jx\| \to \infty \text{ as } \|x\| \to \infty, \quad \text{where } J \text{ and } G \text{ are as above.}
\]

Then $T(X) = Y$.

**Proof.** Let $f$ in $Y$ be fixed. By condition (5) there exists an $r_f > 0$ such that
\[
\|Tx - tf\| > 0 \quad \text{for } \|x\| = r_f, \quad t \in [0, 1]
\]
and
\[
\|Tx\| + \frac{(Tx, Jx)}{\|Jx\|} > 0 \quad \text{for } \|x\| = r_f.
\]
The first inequality implies that
\[
\deg(T - f, B(0, r_f), 0) = \deg(T, B(0, r_f), 0),
\]
which is nonzero by the second inequality as shown in Corollary 1. Hence, $Tx = f$ is solvable. $\qed$

**Remark.** Along similar lines one can show that if $T: X \to X$ is continuous and compact (or condensing) and $I - T$ satisfies condition (5), then $(I - T)(X) = X$ (the proof will appear in a forthcoming paper by the author).

Condition (5) for $PN$ clearly holds if $PN$ is coercive on $X_1$, i.e.,

- if $(PNx, Jx)/\|Jx\| \to \infty$ as $\|x\| \to \infty$, $x \in X_1$, or
- if $(PNx, Jx) \geq -c_1\|Jx\|$ for all $x \in X_1$ and some $c_1 > 0$ and $\|PNx\| \to \infty$ as $\|x\| \to \infty$, $x \in X_1$, and, in particular,
- if $\|PNx\| > c_2\|x\|^k$ for all $x \in X_1$ and some $c_2 > 0$, $k > 0$.

The last condition holds if $N$ is $k$-homogeneous. Indeed, since $\|PNx\| \neq 0$ for $x \in \partial B(0, r) \subset X_1$, $a = \min\{\|PNx\| : \|x\| = r\} > 0$

and $\|PNx\| > (a/r^k)\|x\|^k$ for all $\|x\| > r$.

In view of the above discussion, we have the following special case of Theorem 2.1 in [8]:

**Theorem 4.** Let $A: D(A) \subset X \to Y$ be a linear Fredholm map of index zero and $N: \overline{D} \subset X \to X$ a continuous compact map, where $D$ is open and bounded. Suppose that

(i) $Ax \neq \lambdaNx$ for $x \in D(A) \cap \partial D$ and $\lambda \in (0, 1)$;
(ii) $P N x \neq 0$ for each $x \in \ker A \cap \partial D$;
(iii) for some isomorphism $L: Y_1 \to X_1$,
$$
\|LP N x\| + \frac{(LP N x, J x)}{\|J x\|} > 0 \quad \text{for} \ x \in \partial D \cap X_1
$$
with $J$ the normalized duality map from $X_1$ to $2^{X_1}$.

Then the equation $A x - \lambda N x = 0$ has at least one solution in $D$ for each
$\lambda \in [0, 1]$.

**Proof.** It suffices to show (cf. [8]) that $\deg (LP N|_{X_1}, D \cap X_1, 0) \neq 0$. But, this follows from condition (iii) as in Corollary 1 since $I$ is odd. □

**Remark.** The above results could be proven by using the homotopy
$$
H(t, x) = (x^2 + t H(I - P) N x - t f_2, PN(x_1 + t x_2) - t f_1)
$$
instead. Hence, it is sufficient to require that the map $H(I - P) N: X \to X$ be compact or condensing. The same observation holds for Theorem 2 with $N$ replaced by $B$. Moreover, Theorem 2 of Podolak [11] can be shown to be valid for the nonlinearities considered in our Theorem 2.

**References**


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