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A COMPLETE BOUNDED COMPLEX SUBMANIFOLD OF C^3

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ABSTRACT. We produce an example of a bounded complete complex sub-
manifold of C^3. This is accomplished by using the duality between H^1(T) and BMO(T).

The question of whether there exists a complete bounded complex sub-
manifold of C^n has been an open problem (see [3] for definitions and a
discussion of this problem). We present here a method of producing such
submanifolds. Suppose that f_1(z) and f_2(z) are two functions which are
analytic and bounded in the unit disk Δ of C, and suppose that these two
functions have the property

\[ \int_{Γ} \left[ |f_1(z)| + |f_2(z)| \right] \, ds(Γ) = \infty \]  (1.1)

for all curves Γ ⊂ Δ which terminate on ∂Δ = T. (Here σ denotes Euclidean
arc length.) Then z ∈ Δ → (z, f_1(z), f_2(z)) is an embedded complete bounded
complex submanifold of C^3. To construct two bounded analytic functions f_1
and f_2 satisfying (1.1) we use C. Fefferman's theorem [1] that every real
valued function ϕ ∈ BMO(T) can be represented by

ϕ = u + v, u, v ∈ L^∞(T).

Here v denotes the Hilbert transform of ϕ. Consider the harmonic function

ϕ(r e^{iθ}) = \sum_{n=1}^{∞} \frac{r^{10^n}}{n} \cos 10^nθ.

Then |∇ϕ(z)| > 10^n/100n if z is in the annulus

A_n = \{ z : 1 - 11 \cdot 10^{-n-1} < |z| < 1 - 9 \cdot 10^{-n-1} \}.

To see this, note that |∇((1/n)r^{10^n} \cos 10^nθ)| is of order of magnitude 10^n/n on A_n. The term

\left| \nabla \left( \sum_{j=1}^{n-1} \frac{r^{10^j}}{j} \cos 10^jθ \right) \right|

is small on A_n because it is bounded pointwise by 2Σ_{j=1}^{n-1} 10^j/j. The term

\left| \nabla \left( \sum_{j=n+1}^{∞} \frac{r^{10^j}}{j} \cos 10^jθ \right) \right|

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is small on $A_n$ because it is bounded there by
\[ 2 \sum_{j=n+1}^{\infty} 1/j \cdot 10^j \cdot e^{-(1/2)j10^{-n}}. \]

Now if $\Gamma$ is a curve in $\Delta$ terminating on $T$,
\[ \int_{\Gamma} |\nabla \varphi(z)| \, d\sigma(z) = \infty, \]
because $\Gamma$ must cross $A_n$ for all $n$ larger than some integer. It is easy to check by hand that $\varphi(e^{it}) \in \text{BMO}(T)$. (This is clear anyway by Paley's theorem.) By Fefferman's theorem, $\varphi = u + \delta$ for some $u, \nu \in L^\infty(T)$. Let $f_1 = e^{it + i\delta}$ and $f_2 = e^{it + i\delta}$. Then $f_1$ and $f_2$ are in $H^\infty(\Delta)$, and since $f_1$ and $f_2$ are bounded from below on $\Delta$,
\[ |f_1(z)| + |f_2(z)| > c|\nabla \varphi(z)| \]
for some constant $c$. This means that $f_1$ and $f_2$ satisfy property (1.1).

We note that by replacing $f_1$ by $f_1 + \alpha z$ for a suitable $\alpha \in \mathbb{C}$,
\[ z \in \Delta \rightarrow (f_1(z) + \alpha z, f_2(z)) \]
yields a complete bounded immersed curve in $C^2$. (Just pick $\alpha$ so that $\{ z : f_1(z) = -\alpha \} \cap \{ z : f_2(z) = 0 \} = \emptyset$.)

With only a little more work one can produce a proper holomorphic mapping $\varphi$ from $\Delta$ to the ball in $C^4$ such that the image of $\Delta$ is a complete complex submanifold. Let $\varphi(e^{it})$ be as before. It is easy to check that $\varphi \in \text{VMO}(T)$ (see [2] for the definition of VMO). By a theorem of Sarason [2], $\varphi$ can be represented as $\varphi = u + \delta$, where $u$ and $\nu$ are continuous on $T$. Let $f_1 = ee^{it + i\delta}$ and $f_2 = ee^{it + i\delta}$, where $e$ is chosen so that
\[ 1 - e^2 - |f_1|^2 - |f_2|^2 > \frac{1}{2} \]
on $T$. Let
\[ g(e^{it}) = \frac{1}{2}\log \left( 1 - e^2 - |f_1(e^{it})|^2 - |f_2(e^{it})|^2 \right). \]

Clearly $g$ is continuous on $T$. Put $f_3 = e^{e^{it + i\delta}}$. Then $|f_3(z)|$ is continuous and
\[ e^2 + |f_1(z)|^2 + |f_2(z)|^2 + |f_3(z)|^2 \rightarrow 1 \]
as $|z| \rightarrow 1$. The mapping $\varphi(z) = (ez, f_1(z), f_2(z), f_3(z))$ now does the job.

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