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ON A COMPLETENESS THEOREM OF PALEY AND WIENER

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ABSTRACT. In this brief note we offer a simplified proof of a classical completeness theorem for systems of complex exponential functions $\{e^{i\lambda_n t}\}$.

A sequence of complex exponential functions $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ is said to be *exact* in $L^2[-\pi, \pi]$ if it is complete, that is, if the relations

$$\phi \in L^2[-\pi, \pi] \quad \text{and} \quad \int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t}\, dt = 0 \qquad (-\infty < n < \infty)$$

imply that $\phi(t) = 0$ almost everywhere on $[-\pi, \pi]$, but becomes incomplete upon the removal of a single term.

The following classical theorem of Paley and Wiener [2, p. 89] is fundamental. We offer a simplified version of the proof.

THEOREM. *Let $\{\lambda_n\}_{n=-\infty}^{\infty}$ be a symmetric sequence of real or complex numbers: $\lambda_{-n} = -\lambda_n$ $(n = 0, 1, 2, \ldots)$. If the system $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ is exact in $L^2[-\pi, \pi]$, then the infinite product*

$$f(z) = \prod_{n=1}^{\infty} \left(1 - z^2/\lambda_n^2\right)$$

converges to an entire function of exponential type π for which

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty \quad \textit{and} \quad \int_{-\infty}^{\infty} |xf(x)|^2 = \infty.$$

PROOF. Since $\{e^{i\lambda_n t}\}$ is exact, there exists a function ϕ in $L^2[-\pi, \pi]$ for which

$$\int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t}\, dt = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{1}$$

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Put

$$g(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt}\, dt.$$

Then $g(z)$ is an entire function of exponential type π, square integrable on the real axis, and zero at every λ_n $(n \neq 0)$. Assertion: $g = f$. Observe first that $g(z)$ can vanish only at the λ_n's $(n \neq 0)$. Indeed, suppose that $g(z)$ were zero for some other value, $z = \gamma$ say. It is a simple matter to show that

$$\frac{zg(z)}{z - \gamma} = \int_{-\pi}^{\pi} \phi_1(t) e^{izt}\, dt$$

for a suitable function ϕ_1 in $L^2[-\pi, \pi]$ (see, for example, [1, p. 10]). Since the left side vanishes at *every* λ_n, it follows that the system $\{e^{i\lambda_n t}\}$ is incomplete in $L^2[-\pi, \pi]$, contrary to assumption. Accordingly, $g(z)$ has no zeros other than the λ_n's $(n \neq 0)$, and hence we can write

$$g(z) = e^{Az} \prod_{n=1}^{\infty} (1 - z^2/\lambda_n^2),$$

by virtue of Hadamard's factorization theorem. Since $\{\lambda_n\}$ is symmetric, (1) holds with $\phi(t)$ replaced by $\phi(-t)$. But $\phi(t)$ is uniquely determined, and so must be even. Therefore, $g(z)$ is even, so $A = 0$ and hence $g = f$.

For the second assertion, suppose to the contrary that $xf(x)$ does belong to L^2 along the real axis. By the Paley-Wiener representation,

$$zf(z) = \int_{-\pi}^{\pi} \mu(t) e^{izt}\, dt,$$

where $\mu(t)$ is a nontrivial function in $L^2[-\pi, \pi]$. But $zf(z)$ vanishes at every λ_n. Once again, this contradicts the assumption that $\{e^{i\lambda_n t}\}$ is complete, and the result follows.

REMARK. In [2] it is postulated that the λ_n have density 1 in each half-plane, $x < 0$ and $x > 0$. The above proof avoids this constraint. That such a density must in fact exist whenever $\{e^{i\lambda_n t}\}$ is exact is a consequence of a well-known property of entire functions (see, for example, [1, p. 25]).

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