TRIVIAL EXTENSION OF A RING WITH BALANCED CONDITION

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Abstract. A ring $R$ is called QF-1 if every faithful $R$-module is balanced. In this paper we study commutative QF-1 rings. It is shown that a commutative QF-1 ring is local if and only if it is uniform. It is well known that commutative artinian QF-1 rings are QF, but Osofsky has constructed a nonartinian nonnoetherian commutative injective cogenerator, so QF-1, ring which is a trivial extension of a valuation ring. It is shown that if a trivial extension of a valuation ring is QF-1, then it has a nonzero socle. Furthermore such rings become injective cogenerator rings under certain conditions.

Throughout this paper rings are commutative rings with unity and modules are unital. Hence the trivial extension of a ring $B$ by a $B$-module $E$ is defined to be the ring whose additive group is the direct sum $B \oplus E$ with multiplication given by

$$(b, e) \cdot (b', e') = (bb', b'e + be').$$

An $R$-module $M$ is called balanced if the canonical ring homomorphism of $R$ into the double centralizer of $M$ is surjective, and a ring $R$ is said to be QF-1 if every faithful $R$-module is balanced. A ring $R$ is said to be PF if $R$ is an injective cogenerator as an $R$-module.

There is an interesting theorem on QF-1 rings that a commutative QF-1 artinian ring $R$ is QF (V. P. Camillo [1] and S. E. Dickson and K. R. Fuller [3]) which has been generalized by C. M. Ringel [10], H. H. Storrer [12] and H. Tachikawa [14] under the weaker condition that $R$ is noetherian or perfect instead of the assumption on $R$ to be artinian.

On the other hand B. L. Osofsky [9, Example 1] has constructed a commutative nonnoetherian PF ring which is a trivial extension of a valuation ring. It is known that PF rings are QF-1, so her example distinguishes commutative QF-1 rings from QF rings. Hence trivial extension QF-1 rings of valuation rings are worthy to be considered. The purpose of this paper is to show that such a ring has a nonzero socle, which is an important necessary condition for a ring to be PF, and that it is PF under certain conditions.

V. P. Camillo [2] and H. H. Storrer [11] have proved that a commutative QF-1 ring $R$ has the principal extension property which plays the important
role in this paper, that is, every homomorphism of any principal ideal of $R$ into $R$ can be extended to all of $R$. Furthermore M. Ikeda and T. Nakayama [7] have proved that if a ring $R$ has the principal extension property, then every principal ideal of $R$ satisfies the annihilator condition, that is,

$$\text{Ann}_R \text{Ann}_R(Rr) = Rr \text{ for all } r \in R.$$ 

Although C. M. Ringel has proved in [10] that if a commutative QF-1 ring has a nonzero socle and if it is local then it is uniform, we shall prove it by using the principal extension property without assuming the existence of a nonzero socle and prove the converse of it.

**Theorem 1.** A commutative QF-1 ring is local if and only if it is uniform.

**Proof.** At first we shall consider the case where a local commutative QF-1 ring $R$ has a nontrivial socle. Let $S$ be a minimal ideal. For an arbitrary nonzero element $x$ in $R$, an $R$-module $Rx/(\text{Rad}(R))x$ is simple and isomorphic to $S$. Thus there exists a nonzero $R$-homomorphism of $Rx$ into $R$ whose image is $S$. The principal extension property means $S \subseteq Rx$. So $R$ is uniform since $S$ is essential in $R$.

Next we shall consider a local commutative QF-1 ring $R$ with zero socle. In this case it is to be noted that the maximal ideal $W (= \text{Rad}(R))$ of $R$ is faithful. We shall prove the fact that if $x$ and $y$ are nonzero elements of $R$ with $Rx \cap Ry = 0$, then $\text{Ann}_R(x) + \text{Ann}_R(y) = R$. This implies either $\text{Ann}_R(x) = R$ or $\text{Ann}_R(y) = R$, because $R$ is assumed to be local. Hence either $x$ or $y$ is zero, which contradicts the assumption that $x$ and $y$ are nonzero elements. So suppose that $\text{Ann}_R(x) + \text{Ann}_R(y) \neq R$. Then $W$ contains $\text{Ann}_R(x) + \text{Ann}_R(y)$, and an $R$-module $R/Wx \oplus R/Wy$ is faithful and so balanced, since $Wx \cap Wy \subseteq Rx \cap Ry = 0$. We can define a nonzero map $\phi$ of $R/Wx \oplus R/Wy$ into $R/Wx \oplus R/Wy$ as follows $(a, b) \mapsto (xa, yb)$ for $a \in R/Wx, b \in R/Wy$, since $Rx \neq Wx$ and $Ry \neq Wy$. Every element in $\text{Hom}_R(R/Wx, R/Wy)$ is induced by some element $r$ of $R$ such that $Wxr \subseteq Wy$, so $r$ annihilates $x$ since $Wx \cap Wy = 0$ and $W$ is faithful. Also since $Wyx \subseteq Wy \cap Wx = 0$, we have $\text{Ann}_R(x) \cap Wy$. Consequently we have

$$T(R/Wx, R/Wy) = \text{Ann}_R(x)/Wy,$$

where

$$T(R/Wx, R/Wy) = \sum \{ \text{Im}(f); f \in \text{Hom}_R(R/Wx, R/Wy) \}.$$

Thus $(x - y)T(R/Wx, R/Wy) = 0$ since $\text{Ann}_R(x) \subseteq W$. Similarly $(y - x)T(R/Wy, R/Wx) = 0$. Then by the Camillo criterion [1, 11. Lemma], $\phi$ is an element of the double centralizer of $R/Wx \oplus R/Wy$. So it should be induced by the multiplication with an element $r$ of $R$. But then the element $(1 + Wx, 1 + Wy)$ in $R/Wx \oplus R/Wy$ is mapped onto $(r + Wx, r + Wy) = (x + Wx, y + Wy)$, thus $r - x \in Wx$ and $r - y \in Wy$, and therefore $r \in Rx \cap Ry = 0$, a contradiction.

Conversely, suppose that a commutative QF-1 ring $R$ is uniform. We fix an arbitrary nonunit element $x$ of $R$. For any element $r$ of $R$, the element $rx$ is a
nonunit. If \( \text{Ann}_{R}(rx) = 0 \), a map of \( Rrx \) into \( R \) as follows \( r'rx \mapsto r' \) for \( r' \in R \) is a well-defined \( R \)-homomorphism. By the principal extension property, there is an element \( p \in R \) such that \( prx = 1 \), which contradicts the fact that \( rx \) is a nonunit. Thus we have \( \text{Ann}_{R}(rx) \neq 0 \). Since \( R \) is uniform and \( \text{Ann}_{R}(rx) \cap \text{Ann}_{R}(1 - rx) = 0 \), we have \( \text{Ann}_{R}(1 - rx) = 0 \), so \( 1 - rx \) is a unit by the above argument. We have showed that every nonunit element is in the radical of \( R \), which implies that \( R \) is local.

V. P. Camillo has proved in [1, Lemma 2] that if the direct sum of a faithful \( R \)-module \( M \) and a simple \( R \)-module \( S \) is balanced then either \( \text{Hom}_{R}(S, M) \) or \( \text{Hom}_{R}(M, S) \) is not zero. Hence it follows that if \( R \) is QF-1 and if a faithful \( R \)-module \( M \) has a zero socle, then the (Jacobson) radical \( \text{Rad}(M) \) of \( M \) is not equal to \( M \). We note that for a module \( M \) over a local ring \( R \), \( \text{Rad}(M) \) is equal to \( \text{Rad}(R) \cdot M \).

We need some definitions. A module \( E \) is called uniserial if the lattice of submodules of \( E \) is linearly ordered by inclusion, and a ring \( B \) is called a valuation ring if \( B \) is uniserial as a \( \beta \)-module. A valuation ring \( B \) is said to be maximal if every system of pairwise solvable congruences of the form

\[ x \equiv x_{\alpha}(I_{\alpha}) \quad (\alpha \in A, x_{\alpha} \in B, I_{\alpha} \text{ an ideal of } B) \]

has a simultaneous solution in \( B \). We say \( B \) is almost maximal if the above congruences have a simultaneous solution whenever \( \bigcap_{\alpha \in A} I_{\alpha} \neq 0 \) (cf. C. Faith [4]).

Now, we are ready to prove our next theorem.

**Theorem 2.** Let \( R \) be the trivial extension ring of a valuation ring \( B \) by a nonzero \( B \)-module \( E \). If \( R \) is QF-1, then the following hold:

(a) \( E \) is faithful and uniserial;

(b) the socle of \( R \) is equal to \((0, \text{Soc}_{B}(E))\) and is not zero.

**Proof.** (a) \( R \) is a commutative local ring with the maximal ideal \( \text{Rad}(B), E \), and \( R \) is uniform by Theorem 1. \( (\text{Ann}_{B}(E), 0) \) and \((0, E)\) are ideals with zero intersection and so, by the uniformness of \( R \), \( \text{Ann}_{B}(E) \) must be zero, that is, \( E \) is faithful.

For the second assertion it is sufficient to show that for any two elements \( x, y \) of \( E \), it holds that either \( Bx \subseteq By \) or \( Bx \supsetneq By \). Since \( B \) is a valuation ring, we may assume \( \text{Ann}_{B}(y) \) contains \( \text{Ann}_{B}(x) \). Then

\[ \text{Ann}_{R}(0, x) = (\text{Ann}_{B}(x), E) \subseteq (\text{Ann}_{B}(y), E) = \text{Ann}_{R}(0, y). \]

Two principal ideals \( R(0, x) = (0, Bx) \) and \( R(0, y) = (0, By) \) satisfy the annihilator condition, so we have

\[ (0, Bx) = \text{Ann}_{R}\text{Ann}_{R}(0, x) \supset \text{Ann}_{R}\text{Ann}_{R}(0, y) = (0, By), \]

which shows that \( Bx \supsetneq By \).

(b) Assume that there exists an element \((b, e)\) of \( R \) which generates a minimal ideal such that \( b \neq 0 \). Noting that \( bE \neq 0 \), we have \( R(b, e) \supsetneq (0, bE) \neq 0 \), which contradicts the minimality of \( R(b, e) \). Thus \( \text{Soc}_{R}(R) \) must be \((0, \text{Soc}_{B}(E))\).
Next we must show that $R$ has a nonzero socle. If $\text{Soc}_B(B)$ is not zero, then we have a minimal ideal $(0, Bbe)$, where $b$ is a generator of a minimal ideal of $B$ and $e$ is an element of $E$ such that $be \neq 0$.

Thus we may assume $\text{Soc}_B(B) = 0$. Let $W$ be the radical of $B$. It is to be noted that $W$ is a faithful ideal of $B$.

We claim that $E$ is not a cyclic $B$-module. Suppose $E$ is cyclic with a generator $e$, then the ideal $(W, We)$ of $R$ is faithful and has a zero socle. Then we have $(W, We) \neq \text{Rad}(W, We) = (W^2, We)$. Hence $W$ is generated by one element $w$ since $B$ is a valuation ring. Then

$$(0, We) = \text{Ann}_R \text{Ann}_R(0, we) = \text{Ann}_R(0, E) = (0, E),$$

which contradicts $E \neq We$.

Next, we claim that any proper submodule $F$ of $E$ is not faithful. We can take two elements $e_1, e_2$ of $E$ so that

$$F \subset e_1 \subsetneq Be_2 \subset E.$$  

From the proof of (a), $\text{Ann}_B(e_i)$ contains strictly $\text{Ann}_B(e_2)$; specifically $\text{Ann}_B(e_1)$ is nonzero, so is $\text{Ann}_B(F)$.

Now assume $\text{Soc}_R(R) = 0$. Then the radical $(W, E)$ of $R$ is faithful and has a zero socle, so that $(W, E) \neq \text{Rad}(W, E) = (W^2, WE)$. $WE$ is a faithful $B$-module, so it is equal to $E$. Thus $W$ is generated by one element $w$. $\text{Rad}(R) = R(w, 0)$ leads to $R$ having a nonzero socle by applying the proof of C. M. Ringel [10, Lemma 3], a contradiction. This completes the proof.

**Corollary 3.** Let $R$ be the trivial extension ring of a valuation ring $B$ of an injective nonzero $B$-module $E$. If $R$ is QF-1, then the following hold.

(a) $E$ is the injective hull of $B/\text{Rad}(B)$, so $B$ is an almost maximal valuation ring;

(b) if $E$ is cyclic, then $B$ and $R$ are PF;

(c) if $B$ is not an integral domain, then $R$ is PF.

**Proof.** (a) We know that $(0, \text{Soc}_B(E))$ is minimal and essential in $R$ from Theorems 1 and 2. So $\text{Soc}_B(E)$ is simple and essential in $E$, and thus $E$ is the injective hull of $B/\text{Rad}(B)$. By C. Faith [4, Theorem 20.49], $B$ is almost maximal.

(b) If $E$ is cyclic, $E$ is isomorphic to $B$. By (a), $E$ is an injective cogenerator, so is $B$. By B. J. Müller [8, Theorem 10], then $R$ is PF.

(c) Since $B$ is an almost maximal nonintegral domain, $B$ is maximal by C. Faith [4, Proposition 20.46]. We shall show that the endomorphism ring of $E$ is canonically isomorphic to $B$. This implies that $R$ is injective by R. M. Fossum et al. [6, Corollary 4.37] since $E$ is injective. Now let $f$ be any element of the endomorphism ring of $E$, and $\{e_{ao}\}_{o \in A}$ be a set of generators of $E$. For every $e_{ao}$, a map of $R(0, e_{ao})$ into $R$, as follows $r(0, e_{ao}) \mapsto (0, be_{ao})$ for $r = (b, e) \in R$, is a well-defined $R$-homomorphism, so there exists an element $b_{ao}$ of $B$ such that $fe_{ao} = b_{ao}e_{ao}$ by the principal extension property. Then we consider
the system of congruences as follows:

\[ x \equiv b_\alpha(I_\alpha) \quad (\alpha \in A, b_\alpha \in B, I_\alpha = \text{Ann}_B(e_\alpha)). \]

For any \( \alpha \) and \( \beta \) in \( A \), we may assume that \( Be_\alpha \subset Be_\beta \) since \( E \) is uniserial. If \( e_\alpha = be_\beta \) by an appropriate element \( b \) of \( B \), then \( b_\beta e_\alpha = bb_\beta e_\beta = bfe_\beta = fe_\alpha = b_\alpha e_\alpha \), so \( b_\beta - b_\alpha \in I_\alpha \). This shows that the above system is pairwise solvable. There exists a solution of it, since \( B \) is maximal. This solution induces \( f \). This completes the proof.

C. Faith [5, Theorem 6A] has given equivalent conditions on a trivial extension of a ring to be a PF valuation ring. Here we shall give the necessary and sufficient condition in order that a trivial extension QF-1 ring is a valuation ring.

**Corollary 4.** Let \( R \) be the trivial extension QF-1 ring of a ring \( B \) by a nonzero \( B \)-module \( E \). Then the following are equivalent:

(a) \( R \) is a valuation ring;

(b) \( B \) is an integral domain and is a valuation ring.

**Proof.** We assume (a). For any ideals \( I, J \) of \( B \), \( (I, IE), (J, JE) \) are ideals of \( R \). We have either \( (I, IE) \supset (J, JE) \) or \( (I, IE) \subset (J, JE) \), so either \( I \supset J \) or \( I \subset J \), which means that \( B \) is a valuation ring.

Then \( E \) is faithful by Theorem 2. Next, take any nonzero element \( b \) of \( B \). An ideal of \( R \) generated by \((b, 0)\) is \((Bb, bE)\), which is not contained in an ideal \((0, E)\) of \( R \). Thus it contains \((0, E)\), hence we have \( bE = E \). The regularity of \( b \) follows from the faithfulness of \( E \).

We assume (b). By Theorem 2, \( E \) is uniserial and faithful. We are in the case where \( \text{Soc}_B(B) = 0 \) because \( B \) is an integral domain. From the proof of Theorem 2, any proper submodule of \( E \) is not faithful. Take an ideal \( I \) of \( R \) such that \( I \) contains one element \((b, e)\) with \( b \neq 0 \). Then a submodule \( bE \) of \( E \) is faithful, so equal to \( I \), which leads to \( I \) containing \((0, E)\). Thus the ideals of \( R \) are of the form \((J, E)\), with \( J \) an ideal of \( B \), and of the form \((0, F)\), with \( F \) a submodule of \( E \). Since both \( B \) and \( E \) are uniserial, \( R \) is a valuation ring.

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