PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 1, October 1979

SURFACES IN THE GRASSMANN VARIETY $G(1, 3)$

AIGLI PAPANTONOPOULOU

ABSTRACT. Let G be the Grassmannian $G(1, 3)$, Y a nonsingular subvariety of G and $N_{Y/G}$ the normal bundle of Y in G. Then $N_{Y/G}$ is not an ample bundle if and only if Y is a Schubert cycle.

1. Introduction. In [**8**] we gave a complete characterization of nonsingular curves in any Grassmannian $G(r, n)$ with ample normal bundle. In the case of $G(1, 3)$ we showed that the Schubert cycle of dimension 1, which is a line, is the only nonsingular curve with normal bundle not ample. In this paper, using the intersection theories developed in [**1**] and in [**9**] we show that in dimension 2 also the Schubert cycles, which are isomorphic to $\mathbf{P}^2$, are the only nonsingular surfaces with normal bundle not ample. In the case of codimension 1 the normal bundle is always ample because the $\operatorname{Pic}(G)$ is trivial. Knowing that a nonsingular subvariety of $G(1, 3)$ has an ample normal bundle we can apply on it several well-known theorems: (a) A vanishing theorem on formal schemes [**4**, Theorem 4.1], (b) A theorem on meromorphic functions [**5**, Chapter 6], (c) Results on the cohomological dimension of a projective variety minus a subvariety [**5**, Chapter 7].

Throughout the paper we are working over an algebraically closed field of characteristic 0.

2. Preliminaries. Let $G = G(1, 3)$ be the Grassmannian parametrizing $\mathbf{P}^1$ spaces in a fixed $\mathbf{P}^3$. Then $\dim G = 4$ and G is a quadric in $\mathbf{P}^5$. The Schubert cycles of G can be described as follows:

(a) $Z_0 = G \cap H$, H a hyperplane in $\mathbf{P}^5$, $\dim Z_0 = 3$,

$$Z_0 = \{\mathbf{P}^1 \subset \mathbf{P}^3 | \mathbf{P}^1 \cap \text{some fixed } \mathbf{P}^1 \neq \varnothing\}.$$

(b) $Z_1 = \{\mathbf{P}^1 \subset \mathbf{P}^3 | \mathbf{P}^1 \subset \text{some fixed } \mathbf{P}^1 \subset \mathbf{P}^3\} \cong \mathbf{P}^2$

$$Z_2 = \{\mathbf{P}^1 \subset \mathbf{P}^3 | \text{ some fixed point } p \in \mathbf{P}^1 \subset \mathbf{P}^3\} \cong \mathbf{P}^1.$$

(c) $Z_3 = \{\mathbf{P}^1 \subset \mathbf{P}^3 | \text{ some fixed point } p \in \mathbf{P}^1 \subset \text{ some fixed } \mathbf{P}^2 \subset \mathbf{P}^3\} \cong \mathbf{P}^1$, and for any given Z_3 there are exactly two cycles Z_1 and Z_2 such that $Z_3 = Z_1 \cap Z_2 \subset G$.

Recall that on G we have the short exact sequence

$$0 \to E \to 0_G^4 \to Q \to 0,$$

where Q and E are the canonical bundles on G. The tangent bundle

Received by the editors July 12, 1978 and, in revised form, November 23, 1978.
AMS (MOS) subject classifications (1970). Primary 14M15; Secondary 14J99, 14F05.

$T_G = Q \otimes \check{E}$, therefore for all nonsingular subvarieties of G the normal bundle $N_{Y/G}$ is generated by global sections. Hence by [**2** Proposition 2.1] and [**8**, Chapter 2] if $N_{Y/G}$ is not ample then there exists a curve C in Y and a surjective map

$$N_{Y/G} \to 0_C \to 0.$$

We recall that for Y locally a complete intersection $N_{Y/G}$ is defined as $\mathrm{Hom}_{0_Y}(I_Y/I_Y^2, 0_Y)$, where I_Y is the sheaf of ideals defining Y in G, and when Y is nonsingular $N_{Y/G}$ is simply the usual geometric normal bundle of Y in G.

Finally one more property of G that we need is the fact that the Chow ring $A(G)$ of cycles mod rational equivalence is generated by the Schubert cycles. For example a surface $Y \neq Z_i$, $Y \sim aZ_1 + bZ_2$ where a and b some positive integers, and $Z_1 \cdot Z_2 = 0$, $Z_i^2 = 1$ in $A(G)$.

We use in the proof the definition and properties of ample vector bundles which are shown in [**3**]. We also use repeatedly the following proposition from [**8**]:

PROPOSITION 2.4. *Let Y be a subvariety of $G(r, n)$. Then $T_{G|Y}$ is not ample if and only if there is a curve C in Y which lies in some Z_3.*

In the proof of the theorem we will be using the following notation and diagrams:

Y = a nonsingular surface in $G(1, 3) = G$, Z = a Schubert cycle in G of dim 2, and $D = Y \cap Z$.

G' = the blow up of G with center Y, p: $G' \to G$ the projection,

$Y' = \mathbf{P}(\check{N}_{Y/G})$, Z' = the strict transform of Z, i.e. the blow up of Z with center D, and $D' = Y' \cap Z'$.

$$\begin{tikzcd}
 & D' \arrow[rr] \arrow[dl] \arrow[dd, "\tau"] & & Z' \arrow[dl] \arrow[dd] \\
Y' \arrow[rr] \arrow[dd, "p"'] & & G' \arrow[dd, "p"] & \\
 & D \arrow[rr] \arrow[dl] & & Z \arrow[dl] \\
Y \arrow[rr] & & G &
\end{tikzcd} \tag{2.1}$$

G'' = the blow up of G with center Z, q: $G'' \to G$ the projection,

$Z'' = \mathbf{P}(\check{N}_{Z/G})$, Y'' = the strict transform of Y, i.e. the blow up of Y with center D, and $D'' = Y'' \cap Z'' \cong D'$.

$$\begin{tikzcd}
 & D'' \arrow[rr] \arrow[dl] \arrow[dd, "\tau"] & & Z'' \arrow[dl] \arrow[dd, "q"] \\
Y'' \arrow[rr] \arrow[dd] & & G'' \arrow[dd, "q"] & \\
 & D \arrow[rr] \arrow[dl] & & Z \arrow[dl] \\
Y \arrow[rr] & & G &
\end{tikzcd} \tag{2.2}$$

On Y' we have the canonical sequence

$$0 \to 0_{Y'}(-1) \to p^*(N_{Y/G}) \to F \to 0$$

and restricted to D' we get

$$0 \to N_{D'/Z'} \to p^*(N_{Y/G})_{|D'} \to F_{|D'} \to 0. \tag{1}$$

Also from the canonical sequence on Z'' restricted to D'' we get

$$0 \to N_{D''/Y''} \to q^*(N_{Z/G})_{|D''} \to E_{|D''} \to 0. \tag{2}$$

From [**1**] we have that in the Chow ring $A(D)$ the intersection cycle is given by $Y \cdot Z = \tau_* c_1(F_{|D'}) = \tau_* c_1(D_{|D''})$.

3. Nonsingular surfaces in $G(1, 3)$.

THEOREM. *Let Y be a nonsingular surface in $G = G(1, 3)$. $N_{Y/G}$ the normal bundle of Y in G is not ample if and only if $Y = Z_i$ (a Schubert cycle of* dim 2).

PROOF. If $Y = Z_i$ then $T_{G|Y} = T_Y \oplus Q_{|Y}$ or $T_{G|Y} = T_Y \oplus \check{E}_{|Y}$. By [**8**, 2.2 and 2.3]

$$Q_{|Z_3} \cong \check{E}_{|Z_3} \cong 0_{Z_3}(1) \oplus 0_{Z_3},$$

where $Z_3 \cong \mathbf{P}^1$. Hence $N_{Y/G}$ restricted to a cycle of type Z_3 is not ample, therefore $N_{Y/G}$ is not ample.

Now assume that $Y \neq Z_i$. $N_{Y/G}$ is a vector bundle on Y which is generated by its global sections, hence by [**2**. Proposition 2.1] it is sufficient to show that $N_{Y/G|C}$ is ample for every curve C in Y. By [**8**, Proposition 2.4] $T_{G|C}$ is not ample if and only if $C = Z_3 \cong \mathbf{P}^1$. Since $N_{Y/G|C}$ is a quotient bundle of $T_{G|C}$, it will be sufficient to show that for any cycle $Z \subset G$, $N_{Y/G|D}$ is ample, where $D = Y \cap Z$. Let I_D be the sheaf of ideals defining $D = Y \cap Z$ in Y. Then

$$\tau^*(I_D) \oplus 0_{D''} = \check{N}_{D''/Y''}.$$

For any curve C in D'' not in any fiber the map τ is of degree 1 and $\tau_*(\check{N}_{D''/Y''|C}) = \check{N}_{C/Y}$. For the rest of the proof let C be a cycle of type Z_3. Since $Y \neq Z_i$, $\deg_G Y > 1$ and $(C \cdot C)_Y < 1$. Hence $\tau_*(N_{D''/Y''|C}) = 0_C(v)$ where $v < 1$, and the exact sequence (2) restricted to C becomes

$$0 \to 0_C(v) \to q^*(N_{Z/G})_{|C} \to E_{|C} \to 0.$$

The vector bundle

$$q^*(N_{Z/G})_{|C} = 0_C(1) \oplus 0_C,$$

therefore $E_{|C}$ is an ample line bundle on C. Hence the line bundle $E_{|D''}$ has a global section with zeroes on C. Since $\tau_* c_1(F_{|D'}) = \tau_* c_1(E_{|D''})$ we get a global section of the line bundle $F_{|D'}$ which is not constant on C. Hence $F_{|C}$ is an ample line bundle on C. Consider now the exact sequence (1). $\tau_*(N_{D'/Z'|C}) = N_{C/Z}$ where $C \cong \mathbf{P}^1$ and $Z \cong \mathbf{P}^2$. Hence $\tau_*(N_{D'/Z'|C}) = 0_C(1)$ and $N_{Y/G|C} = 0_C(u) \oplus 0_C(w)$ where $u > 0$ and $w > 0$. Hence $N_{Y/G|D}$ is ample. Q.E.D.

BIBLIOGRAPHY

1. W. Fulton and R. MacPherson, *Intersecting cycles on an algebraic variety*, Aarhus Univ., Aarhus, 1976 (preprint).

2. D. Gieseker, *p-ample bundles and their Chern classes*, Nagoya Math. J. **43** (1971), 91–116.

3. R. Harshorne, *Ample vectors bundles*, Inst. Hautes Études Sci. Publ. Math., No. 29 (1966), 63–94.

4. _____, *Ample subvarieties of algebraic varieties*, Lecture Notes in Math., vol. 156, Springer-Verlag, Berlin and New York, 1970.

5. _____, *Cohomological dimension of algebraic varieties*, Ann. of Math. (2) **88** (1968), 403–450.

6. _____, *Ample vector bundles on curves*, Nagoya Math. J. **43** (1971), 71–89.

7. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*. Vols. I, II, III, Cambridge Univ. Press, 1947, 1952, 1954.

8. A. Papantonopoulou, *Curves in Grassmann varieties*, Nagoya Math. J. **66** (1977), 121–137.

9. _____, *Non-proper intersection of surfaces and applications*, Deltion Greek Math. Soc. **18** (1977), 80–93.

DEPARTMENT OF MATHEMATICS, BRYN MAWR COLLEGE, BRYN MAWR, PENNSYLVANIA 19010