SURFACES IN THE GRASSMANN VARIETY $G(1, 3)$

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Abstract. Let $G$ be the Grassmannian $G(1, 3)$, $Y$ a nonsingular subvariety of $G$ and $N_{Y/G}$ the normal bundle of $Y$ in $G$. Then $N_{Y/G}$ is not an ample bundle if and only if $Y$ is a Schubert cycle.

1. Introduction. In [8] we gave a complete characterization of nonsingular curves in any Grassmannian $G(r, n)$ with ample normal bundle. In the case of $G(1, 3)$ we showed that the Schubert cycle of dimension 1, which is a line, is the only nonsingular curve with normal bundle not ample. In this paper, using the intersection theories developed in [1] and in [9] we show that in dimension 2 also the Schubert cycles, which are isomorphic to $P^2$, are the only nonsingular surfaces with normal bundle not ample. In the case of codimension 1 the normal bundle is always ample because the Pic($G$) is trivial. Knowing that a nonsingular subvariety of $G(1, 3)$ has an ample normal bundle we can apply on it several well-known theorems: (a) A vanishing theorem on formal schemes [4, Theorem 4.1], (b) A theorem on meromorphic functions [5, Chapter 6], (c) Results on the cohomological dimension of a projective variety minus a subvariety [5, Chapter 7].

Throughout the paper we are working over an algebraically closed field of characteristic 0.

2. Preliminaries. Let $G = G(1, 3)$ be the Grassmannian parametrizing $P^1$ spaces in a fixed $P^3$. Then dim $G = 4$ and $G$ is a quadric in $P^5$. The Schubert cycles of $G$ can be described as follows:

(a) $Z_0 = G \cap H$, $H$ a hyperplane in $P^5$, dim $Z_0 = 3$.

$Z_0 = \{P^1 \subset P^3| P^1 \cap \text{some fixed } P^1 \neq \emptyset\}.$

(b) $Z_1 = \{P^1 \subset P^3| P^1 \subset \text{some fixed } P^1 \subset P^3\} \simeq P^2$

$Z_2 = \{P^1 \subset P^3| \text{some fixed point } p \in P^1 \subset P^3\} \simeq P^1.$

(c) $Z_3 = \{P^1 \subset P^3| \text{some fixed point } p \in P^1 \subset \text{some fixed } P^2 \subset P^3\} \simeq P^1$, and for any given $Z_3$ there are exactly two cycles $Z_1$ and $Z_2$ such that $Z_3 = Z_1 \cap Z_2 \subset G$.

Recall that on $G$ we have the short exact sequence

$$0 \to E \to O_G \to Q \to 0,$$

where $Q$ and $E$ are the canonical bundles on $G$. The tangent bundle

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$T_G = Q \otimes \hat{E}$, therefore for all nonsingular subvarieties of $G$ the normal bundle $N_{Y/G}$ is generated by global sections. Hence by [2 Proposition 2.1] and [8, Chapter 2] if $N_{Y/G}$ is not ample then there exists a curve $C$ in $Y$ and a surjective map

$$N_{Y/G} \to 0_C \to 0.$$ 

We recall that for $Y$ locally a complete intersection $N_{Y/G}$ is defined as $\text{Hom}_G(I_Y/I_Y^2, 0_Y)$, where $I_Y$ is the sheaf of ideals defining $Y$ in $G$, and when $Y$ is nonsingular $N_{Y/G}$ is simply the usual geometric normal bundle of $Y$ in $G$.

Finally one more property of $G$ that we need is the fact that the Chow ring $A(G)$ of cycles mod rational equivalence is generated by the Schubert cycles. For example a surface $Y \neq Z$, $Y \sim aZ_1 + bZ_2$ where $a$ and $b$ some positive integers, and $Z_1 \cdot Z_2 = 0$, $Z_1^2 = 1$ in $A(G)$.

We use in the proof the definition and properties of ample vector bundles which are shown in [3]. We also use repeatedly the following proposition from [8]:

**Proposition 2.4.** Let $Y$ be a subvariety of $G(r, n)$. Then $T_{G|Y}$ is not ample if and only if there is a curve $C$ in $Y$ which lies in some $Z_3$.

In the proof of the theorem we will be using the following notation and diagrams:

- $Y$ = a nonsingular surface in $G(1, 3) = G$, $Z$ = a Schubert cycle in $G$ of dim 2, and $D = Y \cap Z$.
- $G'$ = the blow up of $G$ with center $Y$, $p: G' \to G$ the projection,
- $Y' = \text{P}(\tilde{N}_{Y/G})$, $Z' = \text{the strict transform of } Z$, i.e. the blow up of $Z$ with center $D$, and $D' = Y' \cap Z'$.

![](image1.png)

$G'' = \text{the blow up of } G \text{ with center } Z$, $q: G'' \to G$ the projection,
- $Z'' = \text{P}(\tilde{N}_{Z/G})$, $Y'' = \text{the strict transform of } Y$, i.e. the blow up of $Y$ with center $D$, and $D'' = Y'' \cap Z'' \equiv D'$.

![](image2.png)
On $Y'$ we have the canonical sequence
\[ 0 \to 0_Y(-1) \to p^*(N_{Y/G}) \to F \to 0 \]
and restricted to $D'$ we get
\[ 0 \to N_{D'/Z} \to p^*(N_{Y/G})_{|D'} \to F_{|D'} \to 0. \tag{1} \]
Also from the canonical sequence on $Z''$ restricted to $D''$ we get
\[ 0 \to N_{D''/Y''} \to q^*(N_{Z/G})_{|D''} \to E_{|D''} \to 0. \tag{2} \]
From [1] we have that in the Chow ring $A(D)$ the intersection cycle is given by $Y \cdot Z = \tau_*c_1(F_{|D}) = \tau_*c_1(D_{|D'})$.

3. Nonsingular surfaces in $G(1, 3)$.

THEOREM. Let $Y$ be a nonsingular surface in $G = G(1, 3)$. $N_{Y/G}$ the normal bundle of $Y$ in $G$ is not ample if and only if $Y = Z$, (a Schubert cycle of dim 2).

PROOF. If $Y = Z$, then $T_{G|Y} = T_Y \oplus Q_{|Y}$ or $T_{G|Y} = T_Y \oplus E_{|Y}$. By [8, 2.2 and 2.3]
\[ 0|Z_3 = E_{|Z_3} \otimes 0Z_3, \]
where $Z_3 \cong \mathbb{P}^1$. Hence $N_{Y/G}$ restricted to a cycle of type $Z_3$ is not ample, therefore $N_{Y/G}$ is not ample.

Now assume that $Y \neq Z$, $N_{Y/G}$ is a vector bundle on $Y$ which is generated by its global sections, hence by [2, Proposition 2.1] it is sufficient to show that $N_{Y/G|C}$ is ample for every curve $C$ in $Y$. By [8, Proposition 2.4] $T_{G|C}$ is not ample if and only if $C = Z_3 \cong \mathbb{P}^1$. Since $N_{Y/G|C}$ is a quotient bundle of $T_{G|C}$, it will be sufficient to show that for any cycle $Z \subset G$, $N_{Y/G|D}$ is ample, where $D = Y \cap Z$. Let $I_D$ be the sheaf of ideals defining $D = Y \cap Z$ in $Y$. Then
\[ \tau^*(I_D) \oplus 0_D = N_{D'/Y''}, \]

For any curve $C$ in $D''$ not in any fiber the map $\tau$ is of degree 1 and $\tau^*(N_{D'/Y''|C}) = N_{C/Y}$. For the rest of the proof let $C$ be a cycle of type $Z_3$. Since $Y \neq Z$, $\deg_G Y > 1$ and $(C \cdot C)_Y < 1$. Hence $\tau^*(N_{D'/Y''|C}) = 0_C(v)$ where $v < 1$, and the exact sequence (2) restricted to $C$ becomes
\[ 0 \to 0_C(v) \to q^*(N_{Z/G})_{|C} \to E_{|C} \to 0. \]

The vector bundle
\[ q^*(N_{Z/G})_{|C} = 0_C(1) \oplus 0_C, \]
therefore $E_{|C}$ is an ample line bundle on $C$. Hence the line bundle $E_{|D'}$ has a global section with zeroes on $C$. Since $\tau_*c_1(F_{|D}) = \tau_*c_1(E_{|D'})$ we get a global section of the line bundle $F_{|D'}$ which is not constant on $C$. Hence $F_{|C}$ is an ample line bundle on $C$. Consider now the exact sequence (1). $\tau^*(N_{D'/Z'|C}) = N_{C/Z}$ where $C \cong \mathbb{P}^1$ and $Z \cong \mathbb{P}^2$. Hence $\tau^*(N_{D'/Z'|C}) = 0_C(1)$ and $N_{Y/G|C} = 0_C(u) \oplus 0_C(w)$ where $u > 0$ and $w > 0$. Hence $N_{Y/G|D}$ is ample. Q.E.D.
Bibliography


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