ADDENDUM TO “RESIDUAL LINEARITY FOR CERTAIN NILPOTENT GROUPS”

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Abstract. In this note we offer an example of a residually $\mathbb{Q}$-linear, but not residually finite, nilpotent group whose center is $\mathbb{Z} \oplus \mathbb{Z}$.

This work rounds off some things which have been considered in [1].

Probably the best known result on residual finiteness for linear groups is Mal'cev's Theorem which says that a finitely generated linear group, over a commutative field, is residually finite. This result gives, in particular, that for finitely generated groups, residual finiteness and residual linearity are equivalent. Suppose now that $G$ is a nilpotent linear group. Then it is quite easy to prove that $G$ is finitely generated if its center is so [1, Lemma 1]. Hence we see that a nilpotent linear group with finitely generated center is residually finite. Thus it seems sensible to ask: Is any residually linear nilpotent group with finitely generated center always residually finite? For nilpotent groups of class 2 the answer is affirmative [1, Corollary]. However the answer is negative for groups of class 3. In fact, for each prime $p$ there exists a nilpotent group of class 3 with cyclic center, which is residually $K$-linear but not residually finite, where $K$ is a field containing the $p^n$-roots of unity, for all $n > 1$ [1, Theorem I(iii)]. It appears to us that $K$ is exceedingly large and so the above question looks more interesting if we consider $\mathbb{Q}$-linearity. In this direction we prove the following:

**Proposition 1.** (i) Let $G$ be a residually $\mathbb{Q}$-linear nilpotent group with cyclic center. Then $G$ is residually finite.

(ii) There exists a residually $\mathbb{Q}$-linear, but not residually finite, nilpotent group whose center is $\mathbb{Z} \oplus \mathbb{Z}$.

Our notation follows that of [1]. However we need the following. Let $G$ be a group. We say that $G$ is a $C$-group if there exists a finite subset $A$ of $G$ such that $ZX(G) = CG(X)$. If $G/\text{Z_i}(G)$ is a $C$-group, for all $i \geq 0$, we say that $G$ is a $C_0^\infty$-group. For example, any linear group, over a field, is a $C_0$-group. Then we can prove the following:

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Proposition II. Let $G$ be a nilpotent group such that $\langle x \rangle^G$ is finitely generated, for all $x \in G$. Then the following are equivalent:

(i) $G$ is residually center-by-finite,
(ii) $G$ is residually abelian-by-finite,
(iii) $G$ is residually linear,
(iv) $G$ is residually $C$.

Proof of Proposition I. (i) It follows from [3, Ex. 2.0] that $G \cong \prod P_i \times R$, where $P_i$ is a torsion-free $Q$-linear group and $R$ is a residually finite group. Denote the projections of $G$ into $P_i$ and $R$ by $\pi_i$ and $\pi$, respectively. Let $z$ be a generator of $Z_i(G)$. Suppose that $\pi_i(z) \neq 1$, for some $i$. Since $P_i$ is torsion-free we see that $\pi_i$ is a monomorphism. Thus $G$ is finitely generated and so residually finite. Suppose $\pi_i(z) = 1$, for all $i$. Then $G \cong R$ and so $G$ is residually finite.

(ii) Let $p$ be a prime. Let $H$ be a group generated by $u, x_i, y_i, t_i, i = 1, 2, \ldots$, subject to the relations

$$[x_i, x_j] = [y_i, y_j] = [t_i, t_j] = 1, \quad \text{for all } i, j,$$

$$[x_i, y_i] = t_i^p, \quad [x_i, y_j] = [t_i, x_j] = [t_i, y_j] = 1 \quad \text{if } i \neq j, \text{ for all } i,$$

$$[t_i, x_i] = [t_i, y_i] = u^{p^i}, \quad \text{for all } i; \quad u \text{ is central in } H.$$

$H$ is a torsion-free nilpotent group of class 3, with $Z_i(G) = \langle u \rangle$. We will prove that $H$ is residually finite. Set

$$H_m = \langle x_n, y_n, t_n, u^{p^n} \rangle, \quad \text{for } n > m.$$ 

Clearly $H_m \triangleleft G$, and $H_m \cap \langle u \rangle = \langle u^{p^m} \rangle$. Hence $\bigcap_{m \geq 1} H_m = \langle 1 \rangle$. Since $H/H_m$ is finitely generated, it follows that it is residually finite and that $H$ is also. Let us consider the group $G$ generated by $u, z, x_i, y_i, t_i, i = 1, 2, \ldots$, subject to the relations

$$[x_i, x_j] = [y_i, y_j] = [t_i, t_j] = 1, \quad \text{for all } i, j,$$

$$[x_i, y_i] = t_i^p z, \quad [x_i, y_j] = [x_i, t_j] = [y_i, t_j] = 1 \quad \text{if } i \neq j, \text{ for all } i,$$

$$[t_i, x_i] = [t_i, y_i] = u^{p^i}, \quad \text{for all } i; \quad u, z \text{ are central in } G.$$

$G$ is a nilpotent group of class 3 with $Z_i(G) = \langle u, z \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. We will prove that $G$ is residually $Q$-linear but not residually finite. Let $x \mapsto \bar{x}$ be a homomorphism of $G$ into a finite group $\overline{G}$. Set $|\overline{G}| = m$. Since $\overline{G}$ is finite, there exist integers $r \neq s$ such that $\overline{x}_{rm} = \overline{x}_{sm}$. Then we get

$$\overline{1} = [\overline{x}_{rm}, \overline{y}_{sm}] = [\overline{x}_{sm}, \overline{y}_{sm}] = i_{sm}^{x_{rm}^m} \overline{z} = \overline{z}.$$ 

Therefore $z \in R(G)$. Since $G/\langle z \rangle \cong H$, by the result of the preceding paragraph we obtain that $R(G) \subseteq \langle z \rangle$. Thus $R(G) = \langle z \rangle$. Put $F = \langle u, x_i, y_i \rangle$, for all $i$. Clearly $F \triangleleft G$. Denote the isolator of $F$ by $\sqrt{F}$. It is easily seen that $\sqrt{F} \cap R(G) = \langle 1 \rangle$. It follows from the relations of $G$ that $G/\sqrt{F} \cong Q^+$. Therefore $G$ is residually $Q$-linear and the result follows.
In order to prove Proposition II we use the following results:

**Lemma 1.** Let $G$ be a $C_0$-group. If $H$ is a normal subgroup of $G$ such that $H \cap Z_i(G)$ is finitely generated, then $H \cap Z_i(G)$ is finitely generated, for all $i > 1$.

**Proof.** It is an obvious modification of [1, Lemma 1].

**Lemma 2.** Let $G$ be a $C$-group each of whose finitely generated subgroups satisfy the maximal condition on subgroups. Consider the following possibilities of $G$:

(i) $\Gamma_2(G)$ is finitely generated,
(ii) if $x \in G$, then $\langle x \rangle^G$ is finitely generated,
(iii) $Z_i(G)/Z_1(G)$ is finitely generated for all $i > 1$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If in addition $G$ is nilpotent, then (iii) $\Rightarrow$ (i).

**Proof.** Since $\langle x \rangle^G \subseteq \langle x \rangle \Gamma_2(G)$, we see that (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (iii). First we observe that $G$ is a $C_0$-group. Since homomorphic images of $G$ satisfy (ii) it suffices to show that $G/Z_1(G)$ is a $C$-group. Let $X$ be a finitely generated subgroup of $G$ with $Z_i(G) = C_G(X)$. It follows from the hypothesis that $\langle X \rangle^G$ is finitely generated. Since $[X, G] \subseteq \langle X \rangle^G$ and the finitely generated subgroups of $G$ satisfy the maximal condition on subgroups we see that $[X, G]$ is finitely generated. Suppose that $[x_1, g_1], \ldots, [x_n, g_n]$ are its generators, where $x_i \in X$ and $g_i \in G$. Set

$$H = \langle X, g_1, g_2, \ldots, g_n \rangle.$$ 

Clearly $[X, G] \subseteq [H, H]$. Set

$$N = \{ a \in G | [a, H] \subseteq Z_1(G) \}.$$

The result follows if we prove that $Z_2(G) = N$. Trivially $Z_2(G) \subseteq N$. Moreover $[H, N, H] = [N, H, H] = \langle 1 \rangle$, so P. Hall's lemma [2, Lemma 1.11] yields $[H, H, N] = \langle 1 \rangle$ and hence $[X, G, N] = \langle 1 \rangle$. Since $X \subseteq H$, we have $[N, X, G] = \langle 1 \rangle$. Thus $[G, N, X] = \langle 1 \rangle$, and $[G, N] \subseteq Z_1(G)$. This proves that $N \subseteq Z_2(G)$. The homomorphism $Z_2(G) \rightarrow \langle X \rangle^G \times \cdots \times \langle X \rangle^G$ in which

$$x \mapsto ([x, x_1], \ldots, [x, x_n]),$$

where $X = \langle x_1, x_2, \ldots, x_n \rangle$, proves that $Z_1(G/Z_1(G))$ is finitely generated. Since $G/Z_1(G)$ is a $C_0$-group, the result follows from Lemma 1. If we assume $G$ nilpotent, then [2, Corollary 3.19] proves that (iii) $\Rightarrow$ (i).

**Lemma 3.** Let $G$ be a nilpotent group with finite commutator and $G/Z_1(G)$ finitely generated. Then $G/Z_1(G)$ is finite.

**Proof.** Since $G/Z_1(G)$ is finitely generated, $Z_2(G)/Z_1(G)$ is isomorphic to a subgroup of a finite product $\prod \Gamma_2(G)$. Since finitely generated nilpotent groups with finite center are finite, the result follows.
Proof of Proposition II. Trivially (i) ⇒ (ii) ⇒ (iii) ⇒ (iv). (iv) ⇒ (i).
Clearly we may assume that $G$ is a $C$-group. Lemma 2 yields that both $\Gamma_2(G)$
and $G/Z_1(G)$ are finitely generated. For any integer $n > 1$, denote $G/\Gamma_2(G)^n$
by $G_n$. Then $\Gamma_2(G_n)$ is finite and $G_n/Z_1(G_n)$ is finitely generated. It follows
from Lemma 3 that $G_n$ is center-by-finite. Let $H_n$ be the complete inverse
image of $Z_1(G_n)$ in $G$. $H_n$ is of finite index in $G$ and $[H_n, G] \subseteq \Gamma_2(G)^n$. Since
$\Gamma_2(G)$ is finitely generated we have $\bigcap_{n > 1} \Gamma_2(G)^n = \langle 1 \rangle$. Therefore
$\bigcap_{n > 1} [H_n, G] = \langle 1 \rangle$. The result follows.

References
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