PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 1, October 1979

A NEGATIVE ANSWER TO THE PRIME SEQUENCE QUESTION

RAYMOND C. HEITMANN

ABSTRACT. If P is a complete intersection prime, i.e., a prime ideal generated by $ht(P)$ elements, in a Noetherian domain, can P be generated by a prime sequence, a regular sequence whose initial segments generate prime ideals? The purpose of this article is to present an example showing that this question, the Prime Sequence Question, has a negative answer. The example is a two-dimensional integrally closed domain with a height two complete intersection prime which contains no prime elements.

The prime sequence question was first raised publicly by J. Ohm in [Math. Reviews **54** #5206]. It is discussed in [**1**] and it is shown to have an affirmative answer for certain types of rings. However, the prevailing belief was that the general case would have a negative answer and here we shall demonstrate precisely that. The example given here is closely based on the construction in [**3**]. In light of the large number of alterations which are necessary, it seems preferable to make the construction here independent of the earlier construction. Consequently, familiarity with [**3**] will not be necessary.

NOTATION. Let K denote a fixed countable field. Let X_1, X_2, $\{Y_i\}_{i=1}^{\infty}$ be indeterminates. The subscript will be deleted when we wish to denote a family of indeterminates, e.g., $Y = \{Y_i\}$. Fix a set A of irreducible polynomials of $K[X, Y]$, choosing precisely one generator for each height one prime of $K[X, Y]$ contained in the ideal (X_1, X_2).

CONSTRUCTION. Since A is countable, we may enumerate its elements $f_1, f_2, \ldots$. Moreover, it is easy to do this in such a way that $f_k \in K[X_1, X_2, Y_1, \ldots, Y_{k-1}]$. Using this, define new elements $Z_k = f_k^2/Y_k$. Next we define a family of rings.

For $k > 0$, set $R_k = (K(Y - \{Y_k\}, Z_k)[X])_{(f_k)}$, the localization of a polynomial ring at a height one prime and consequently a discrete valuation ring. We let v_k denote the appropriate valuation. Note that our method of indexing the elements of A guarantees that R_k is well defined. For $k = 0$, set $R_0 = (K(Y)[X])_S$ where S is the multiplicative set consisting of all products of elements in

Received by the editors July 29, 1978 and, in revised form, November 7, 1978.
AMS (MOS) subject classifications (1970). Primary 13A15, 13F99.
Key words and phrases. Noetherian ring, prime sequence, complete intersection prime, valuation.

$$S' = \left\{ Y_m^{n+1} + \sum_{i=0}^{n} a_i Y_m^i \;\middle|\; \begin{array}{l} a_i \in K[X, Y - \{Y_m\}] \\ v_m(a_n) = 0 \\ v_m(\Sigma a_i Y_m^i) = 2n \end{array} \right\} \cup \{ Y_m^3 + f_m^4 \}.$$

Finally, set $R = \bigcap_{k=0}^{\infty} R_k$. We shall show that R is the desired example.

LEMMA 1. $K[X, Y, Z] \subset R$.

PROOF. It suffices to show $K[X, Y, Z] \subset R_k$ for each k. Obviously, $K[X] \subset R_k$; likewise, for $i \neq k$, $Y_i \in R_k$ and $Z_k \in R_k$. Further, since $f_k \in R_k$ and Z_k is a unit of R_k, $Y_k \in R_k$. Thus, $Y \subset R_k$ and so, for each $i, f_i \in R_k$. When $i \neq k$, Y_i is a unit of R_k and therefore $Z_i \in R_k$.

LEMMA 2. *For each $k \geqslant 0$, R_k is a localization of R and so a flat algebra over R.*

PROOF. The first half of the statement is immediate from the definition of R_k and Lemma 1. Further, localizations are always flat algebras.

LEMMA 3. *Every element of R is a nonunit in at most finitely many R_k.*

PROOF. Since $K[X, Y] \subset R \subset K(X, Y)$, it suffices to consider elements of $K[X, Y]$. Such an element g involves only finitely many indeterminates and is divisible, in the polynomial ring $K[X, Y]$, by only finitely many f_i. Hence, there exists an N such that $g \in K[Y_1, \ldots, Y_N, X_1, X_2]$ and $g \notin f_k K[X, Y]$ for all $k > N$. Then $g \notin f_k K[Y - \{Y_k\}, Z_k, X]$ and hence also $\notin f_k K(Y - \{Y_k\}, Z_k)[X]$; so g is a unit of R_k for all $k > N$.

LEMMA 4. *For any nonzero prime P of R, there exists a k and a prime Q of R_k so that $P = Q \cap R$. Clearly then, $PR_k \neq R_k$.*

PROOF. First we assert that $P = \bigcup (P_{ki} \cap R)$ where P_{ki} is a nonzero prime of R_k and the union is taken over all $P_{ki} \cap R \subset P$. The verification is straightforward and we shall not include it here. In fact, precisely this item is shown in [3, p. 671] in the first paragraph of the proof of Lemma 4.

Next we claim that only a single k occurs in the representation $P = \bigcup (P_{ki} \cap R)$. If the claim is valid, P contains no units of R_k and so $PR_k \neq R_k$. Inasmuch as R_k is a localization of R and there is a one-to-one correspondence between primes of R_k and primes of R which do not blow up in R_k, P must correspond to a prime Q of R_k, i.e., $P = Q \cap R$. So it suffices to verify the claim, which we shall do by contradiction.

Suppose $P_{mi} \cap R$, $P_{ni} \cap R \subset P$ where $m \neq n$. There are two cases to consider: either both m, n are nonzero or one of them, say n, is zero. In the first case, note that P_{mi} must be the unique nonzero prime of R_m. Therefore, $Y_m = (1/Z_m)f_m^2 \in P_{mi}$. Thus, Y_m (and similarly Y_n) is in P. This yields $(Y_m + Y_n) \in P$. However, the latter element is a unit in each R_k and so in R as well–a contradiction.

Therefore, we may assume $P = (P_m \cap R) \cup (\bigcup (P_{0i} \cap R))$. Consider the subring $T = K[X, Y, Z_m] \subset R$. Note that R_m is a localization of T and so $R_m = T_{(P_m \cap T)}$, and also note that $P_m \cap R \subsetneqq P$ implies $R_P \subsetneqq R_m$. Putting these facts together, we obtain $T_{P \cap T} \subset R_P \subsetneqq R_m = T_{(P_m \cap T)}$ and so $P_m \cap T \subsetneqq P \cap T$. This yields an element $g \in P \cap T - P_m \cap T$. To continue the proof, we require g to satisfy certain additional properties which this particular element may not. However, whenever $g' \in P_m \cap T$, $(g - g') \in P \cap T - P_m \cap T$ and so we may safely replace g by $(g - g')$. We proceed thus. First note that g is a sum of monomials in T and those monomials containing Y_m are in P_m. Delete them, leaving $g \in K[X, Y - \{Y_m\}, Z_m]$. Next express g as a polynomial in Z_m with coefficients in $K[X, Y - \{Y_m\}]$. As before, if any coefficient is in P_m, delete the appropriate term.

If $Z_m \notin P$, we may assume (dividing if necessary) that g has nonzero constant term. Thus $g = a_0 Z_m^n + \cdots + a_n$ with $v_m(a_n) = 0$. Set $h = Y_m^n g = a_n Y_m^n + \cdots + a_0 f_m^{2n}$ and observe the conditions we have forced upon g are precisely those needed to conclude $Y_m^{n+1} + h \in S'$. Thus $Y_m + g = (1/Y_m^n)(Y_m^{n+1} + h)$ is a unit of R_0. But it is also a unit of R_m and so cannot be in P, contradicting $Y_m, g \in P$.

On the other hand, if $Z_m \in P$, then $Y_m + Z_m^2 \in P$. However, $Y_m + Z_m^2 = (1/Y_m^2)(Y_m^3 + f_m^4)$ is a unit in both R_0 and R_m. This contradiction completes the proof of the claim and hence the lemma.

Now, according to a theorem of Heinzer and Ohm [**2**, Corollary 1.8, p. 295], whenever R is the intersection of a family $\{R_\alpha\}$ of Noetherian flat R-algebras such that each ideal $I \subset R$ satisfies $IR_\alpha \neq R_\alpha$ for at least one and at most finitely many R_α, then R must be Noetherian. The hypothesis of this theorem has been verified by Lemmas 2, 3 and 4. Hence

THEOREM 5. *R is Noetherian.*

LEMMA 6. *If u is a unit of R_0, then $v_k(u)$ is even for every $k > 0$.*

PROOF. R_0 is a localization of $K[X, Y]$ at the nonzero elements of $(K[Y])S$. Consequently, it is enough to check the lemma for elements of $K[Y]$ and for factors of elements in S'. We claim that for each k, $P_k \cap K[X, Y] = (f_k, Y_k) \subset (X, Y_k)$. To see this, first note $P_k \cap K[X, Y - \{Y_k\}] = (f_k)$. Also $Y_k \in P_k$ as before. Finally, (f_k, Y_k) is the unique prime of $K[X, Y]$ containing Y_k which lies over $f_k K[X, Y - \{Y_k\}]$.

If $u \in K[Y]$, $u = Y_k^m f$ where f is not divisible by Y_k. Clearly $f \notin (X, Y_k)$ and so $v_k(u) = 2m$.

If $s \in S'$ and $k \neq m$ (used in defining s), $s \notin (X, Y_k)$ and so $v_k(s) = 0$. Thus, if u is a factor of s, $v_k(u) = 0$ as well. It only remains to consider $v_m(u)$.

Suppose $s = Y_m^{n+1} + \sum_{i=0}^n a_i Y_m^i$ and $s = uw$. Note $v_m(s) = 2n$. We may assume, factoring as polynomials in Y_m, that $u = Y_m^d + b_{d-1} Y_m^{d-1} + \ldots$ and $w = Y_m^e + c_{e-1} Y_m^{e-1} + \ldots$. Then $d + e = n + 1$ and $a_n = b_{d-1} + c_{e-1}$. As $v_m(a_n) = 0$, we may conclude that one of its summands has valuation zero, say $v_m(b_{d-1}) = 0$. Now $v_m(u)$ is in fact $\min\{v_m(b_i Y_m^i)\}$. This can be seen by noting $v_m(u) = v_m(Z_m^d u)$. Then $Z_m^d u$ is a polynomial in Z_m with coefficients in $K[X, Y - \{Y_m\}]$ and f_m divides such a polynomial if and only if it divides each coefficient. Thus $v_m(Z_m^d u)$ will be the minimum of the valuations of the coefficients, which are $\{b_i f_m^{2i}\}$ and $v_m(u)$ will be as claimed. Then $v_m(u) \leqslant v_m(b_{d-1} Y_m^{d-1}) = 2(d-1)$ and $v_m(w) \leqslant v_m(Y_m^e) = 2e$. As $v_m(u) + v_m(w) = 2n = 2(d-1) + 2e$, we see that both u and w have even valuations.

Finally, since f_m^4 is not a cube, $s = Y_m^3 + f_m^4$ is irreducible. So we need only consider $v_m(s) = 4$. This completes the proof.

THEOREM 7. *The ideal $(X_1, X_2)R$ is a height two prime and so has a regular system of parameters. However, it contains no principal height one primes and so does not have a prime sequence.*

PROOF. To prove $I = (X_1, X_2)R$ is a height two prime, we need only show $I = (X_1, X_2)R_0 \cap R$. By [2, 1.7], it suffices to show $IR_k = ((X_1, X_2)R_0 \cap R)R_k$ for each k. However, for $k > 0$, $IR_k = R_k = ((X_1, X_2)R_0 \cap R)R_k$. Also $IR_0 = (X_1, X_2)R_0 = ((X_1, X_2)R_0 \cap R)R_0$. So the first sentence has been shown.

Let Q be any height one prime contained in I. Then $Q = f_j R_0 \cap R$ for some f_j. If $Q = hR$, then $hR_0 = QR_0 = f_j R_0$ and so $h = f_j u$ where u is a unit in R_0. However, $v_j(h) = v_j(f_j) + v_j(u) = 1 + v_j(u)$ is odd and consequently nonzero. Since $h \in R$, $v_j(h) \geqslant 0$ and so $v_j(h) > 0$, i.e., $h \in P_j \cap R$. But $Q \not\subseteq P_j \cap R$ and so Q cannot be principal.

REFERENCES

1. E. Davis, *Prime ideals and prime sequences in polynomial rings*, Proc. Amer. Math. Soc. **72** (1978), 33–38.

2. W. Heinzer and J. Ohm, *Noetherian intersections of integral domains*, Trans. Amer. Math. Soc. **167** (1972), 291–308.

3. R. Heitmann, *Prime ideal posets in Noetherian rings*, Rocky Mountain J. Math. **7** (1977), 667–673.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712