FUNCTION SPACE FLOW INVARIANCE FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF RETARDED TYPE

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Abstract. Let $\mathcal{C}$ denote the Banach space of continuous functions $\phi: [-r, 0] \to \mathbb{R}^n$, let $\Omega \subset \mathcal{C}$ be closed, and let $f: [0, \infty) \times \Omega \to \mathbb{R}^n$ be continuous. In this note we establish necessary and sufficient conditions for function space flow invariance for the functional differential equation:

$$x'(t) = f(t, x_t) \quad \text{for } t > 0, \quad x_0 = \phi \in \Omega.$$ 

That is, for each $\phi \in \Omega$ there exist $b > 0$ and a solution $x: [-r, b] \to \mathbb{R}^n$ such that $x_t \in \Omega$ for each $t \in [0, b]$.

Let $r > 0$ and let $\mathcal{C} = \mathcal{C}([-r, 0]; \mathbb{R}^n)$ denote the Banach space of continuous functions $\phi: [-r, 0] \to \mathbb{R}^n$ with norm $||\phi|| = \max\{|\phi(t)|: t \in [-r, 0]\}$ and let $\Omega \subset \mathcal{C}$ be closed. If $b > 0$ and $x: [-r, b] \to \mathbb{R}^n$ is continuous, we define $x_t \in \mathcal{C}$ by $x_t(\theta) = x(t + \theta)$ for each $t \in [0, b]$. Let $f: [0, \infty) \times \Omega \to \mathbb{R}^n$ be continuous. In this note we establish necessary and sufficient conditions such that for each $\phi \in \Omega$ there exist $b > 0$ and a continuous function $x: [-r, b] \to \mathbb{R}^n$ satisfying the functional differential equation:

$$x'(t) = f(t, x_t) \quad \text{for } t \in [0, b],$$

$$x_0 = \phi,$$

with $x_t \in \Omega$ for $t \in [0, b]$. Our approach is to construct approximate solutions to a related integral equation employing a modification of the techniques found in Martin [4] and Webb [7]. This function space flow invariance is in the spirit of that defined by Hale [1] as opposed to that investigated by authors such as Leela and Moauro [2] and Seifert [6]. Actually, the invariance considered in [2] and [6] is a special case of that defined above. In remarks following the theorem we indicate the relationship between our results and those of [2] and [6]. Results when $\mathbb{R}^n$ is replaced by a general Banach space are also indicated.

Let $\mathcal{C}$, $\Omega$, and $f$ be as above and $t_0 \in [0, \infty)$. We say that

$$x'(t) = f(t, x_t), \quad t > t_0,$$

$$x_{t_0} = \phi \in \Omega,$$

has a local solution provided there exist $b > 0$ and a continuous function $x: [-r + t_0, t_0 + t + b] \to \mathbb{R}^n$ which satisfies (*) with $x_t \in \Omega$ for $t \in [t_0, t_0 + b]$.

We will frequently refer to the functions $F_h: [0, \infty) \times \mathcal{C} \to \mathcal{C}$ and $T(h)$:

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\[ F_h(t, \phi) \theta = \begin{cases} 
\phi(h + \theta), & -r < \theta < -h, \\
\phi(0) + (h + \theta)f(t, \phi), & -h < \theta < 0, 
\end{cases} \]

and

\[ [T(h)\phi] \theta = \begin{cases} 
\phi(h + \theta), & -r < \theta < -h, \\
\phi(0), & -h < \theta < 0. 
\end{cases} \]

Also, for \( \psi \in \mathcal{C} \) we define \( |\psi|; \Omega = \inf\{||\psi - \phi|| : \phi \in \Omega\} \).

**Theorem.** Suppose \( \Omega \subset \mathcal{C} \) is closed and \( f: [0, \infty) \times \mathcal{C} \to \mathbb{R}^n \) is continuous. Then the following are equivalent:

1. For each \((t_0, \phi) \in [0, \infty) \times \Omega\), \((\ast)\) has a local solution;
2. For each \((t_0, \phi) \in [0, \infty) \times \Omega\), \( \lim_{h \to 0^+} |F_h(t_0, \phi); \Omega|/h = 0 \).

**Proof.** The proof rests on the observation that \( x: [-r + t_0, t_0 + b] \to \mathbb{R}^n \) is a solution to \((\ast)\) if and only if \( x \) is continuous and for each \( t \in [t_0, t_0 + b] \), \( x_\tau \) satisfies

\[
x_\tau(\theta) = \begin{cases} 
\phi(t - t_0 + \theta), & -r < \theta < -t, \\
\phi(0) + \int_{t_0}^{t + \theta} f(\tau, x_\tau) d\tau, & t_0 - t < \theta < 0. 
\end{cases}
\]

That \((1) \Rightarrow (2)\) is now clear; for if \((t_0, \phi) \in [0, \infty) \times \Omega\) and \((\ast)\) has a local solution \( x: [-r + t_0, t_0 + b] \to \mathbb{R}^n \), then

\[
\lim_{h \to 0^+} |F_h(t_0, \phi); \Omega|/h \leq \lim_{h \to 0^+} |F_h(t_0, \phi) - x_{t_0 + \theta}|/h
\]

\[
= \lim_{h \to 0^+} \max_{h < \theta < 0} \left| \phi(0) + (h + \theta)f(t_0, \phi) - \phi(0) \right. \\
\left. - \int_{t_0}^{t_0 + h + \theta} f(\tau, x_\tau) d\tau \right| /h
\]

\[
\leq \lim_{h \to 0^+} \max_{h < \theta < 0} \frac{1}{h} \int_{t_0}^{t_0 + h + \theta} |f(t_0, \phi) - f(\tau, x_\tau)| d\tau
\]

\[
= 0.
\]

The proof that \((2) \Rightarrow (1)\) given below is obtained via approximate solutions to \((\ast \ast)\). The construction of the approximate solutions given in the lemma is patterned after that found in Martin [4] and Webb [7].

Let \( \phi \in \Omega \), and for simplicity we will suppose \( t_0 = 0 \). Since \( f \) is continuous, there exist positive numbers \( b, R, \) and \( M \) such that if \((t, \psi) \in [0, b] \times \Omega\) with \( |\psi - \phi| < R \) then \( |f(t, \psi)| < M \) and if \( |\psi| < b(M + 1) \) then \( |T(t)\phi - \phi + \psi| < R \) for all \( t \in [0, b] \).

**Lemma.** Let \( \Omega \subset \mathcal{C} \) be closed, \( f: [0, \infty) \times \Omega \to \mathbb{R}^n \) be continuous, and \( \lim \inf_{h \to 0^+} |F_h(t, \psi); \Omega|/h = 0 \) for each \((t, \psi) \in [0, \infty) \times \Omega\). Suppose \( \phi \in \Omega \) and \( b, R, \) and \( M \) are as above. Let \( \{\epsilon_n\}^\infty_{n=1} \subset (0, 1) \) with \( \lim_{n \to \infty} \epsilon_n = 0 \). Then for each \( n \) there exist \( U_n: [0, b] \to \mathcal{C} \) and a partition \( \{t_i^n\}^N_{i=0} \) of \([0, b] \) with
$t_{n+1}^n - t_n^n < \varepsilon_n$ such that:

(i) $U_n(0) = \phi$, $U_n(t_n^n) \in \Omega$ with $|U_n(t_n^n) - \phi| < R$ and if $t \in [t_n^n, t_{n+1}^n)$ then $U_n(t)$ is defined by

$$
[U_n(t)]n = \begin{cases} 
\left[ U_n(t_n^n) \right] (t - t_n^n + \theta), & -r \leq \theta < t_n^n - t, \\
\left[ U_n(t_n^n) \right] 0 + (t - t_n^n + \theta) f(t_n^n, U_n(t_n^n)), & t_n^n - t < \theta < 0;
\end{cases}
$$

(ii) $|U_n(t_{n+1}^n) - U_n(t_n^n)| < \varepsilon_n(t_{n+1}^n - t_n^n)$, where $U_n(t_{n+1}^n) = \lim_{t \to t_{n+1}^n} U_n(t)$;

(iii) if $|\theta - \lambda| < t_{n+1}^n - t_n^n$ then $\left[ U_n(t_n^n) \right] \theta - \left[ U_n(t_n^n) \right] \lambda < \varepsilon_n$;

(iv) if $t \in [t_n^n, t_{n+1}^n)$ then $|W_n(t) - U_n(t)| < \varepsilon_n t_n^n$, where for $t \in [t_n^n, t_{n+1}^n)$ we define $W_n(t) \in \mathcal{C}$ by

$$
[W_n(t)]n = \begin{cases} 
\phi(t + \theta), & -r < \theta < -t, \\
\phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & -t < \theta < 0,
\end{cases}
$$

and $\gamma_n(\tau) = t_k^n$ when $\tau \in [t_k^n, t_{k+1}^n)$;

(v) if $(t, \psi) \in [t_n^n, t_{n+1}^n] \times \Omega$ with

$$
|\psi - U_n(t_n^n)| \leq (t_{n+1}^n - t_n^n)(M + 1) + \max\{|T(h)U_n(t_n^n) - U_n(t_n^n)|: 0 < h < t_{n+1}^n - t_n^n\}
$$

then

$$
|f(t, \psi) - f(t_n^n, U_n(t_n^n))| < \varepsilon_n \quad \text{and} \quad |T(t - t_n^n)U_n(t_n^n) - U_n(t_n^n)| < \varepsilon_n.
$$

**Proof.** The construction of $U_n$ is by induction; in particular, we suppose $U_n$ is defined on $[0, t_n^n]$. Let $\delta_n^n$ be the maximum number satisfying

(a) $\delta_n^n \in [0, \varepsilon_n]$ and if $(t, \psi) \in [t_n^n, t_{n+1}^n + \delta_n^n] \times \Omega$ with

$$
|\psi - U_n(t_n^n)| \leq \delta_n^n(M + 1) + \max\{|T(h)U_n(t_n^n) - U_n(t_n^n)|: 0 < h < \delta_n^n\}
$$

then

$$
|f(t, \psi) - f(t_n^n, U_n(t_n^n))| < \varepsilon_n
$$

and

$$
\max\{|T(h)U_n(t_n^n) - U_n(t_n^n)|: 0 < h < \delta_n^n\} < \varepsilon_n;
$$

(b) if $|\theta - \lambda| < \delta_n^n$ then $|U_n(t_n^n) \theta - U_n(t_n^n) \lambda| < \varepsilon_n$; and

(c) $\|F_{\delta_n^n}(t_n^n, U_n(t_n^n))\| < \delta_n^n \varepsilon_n/2$.

The suppositions on $f$ and the definition of $T(h)$ assure that $\delta_n^n > 0$. Let $t_{n+1}^n = t_n^n + \delta_n^n$ and for $t \in [t_n^n, t_{n+1}^n)$ define $U_n(t) \in \mathcal{C}$ by

$$
[U_n(t)]n = \begin{cases} 
\left[ U_n(t_n^n) \right] (t - t_n^n + \theta), & -r \leq \theta < t_n^n - t, \\
\left[ U_n(t_n^n) \right] 0 + (t - t_n^n + \theta) f(t_n^n, U_n(t_n^n)), & t_n^n - t < \theta < 0;
\end{cases}
$$

By (c), $|U_n(t_{n+1}^n)| \leq \delta_n^n \varepsilon_n/2$ so we may choose $U_n(t_{n+1}^n) \in \Omega$ such that

$$
|U_n(t_{n+1}^n) - U_n(t_{n+1}^n)| < (t_{n+1}^n - t_n^n) \varepsilon_n.
$$

To establish that $U_n$ satisfies (i)–(v) on $[0, t_{n+1}^n]$ we first show that $|W_n(t) - W_n(t_{n+1}^n)| < \varepsilon_n t_{n+1}^n$. No
$U_n(t) < \varepsilon_n t^n$ for $t \in [t^n_i, t^n_{i+1})$. Observe that, for $-r < \theta < t^n_i - t$,

$$T(t - t^n_i) W_n(t^n_i) \theta = W_n(t^n_i)(t - t^n_i + \theta)$$

$$\begin{cases} 
\phi(t + \theta), 
&{-r < \theta < -t,} \\
\phi(0) + \int_0^{t^n_i + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau, 
&{-t < \theta < 0,}
\end{cases}$$

and consequently,

$$\max_{-r < \theta < t^n_i - t} |W_n(t)\theta - U_n(t)\theta|$$

$$\begin{aligned}
&= \max_{-r < \theta < t^n_i - t} |T(t - t^n_i) W_n(t^n_i) \theta - T(t - t^n_i) U_n(t^n_i) \theta| \\
&< |W_n(t^n_i) - U_n(t^n_i)| \\
&< |W_n(t^n_i) - U_n(t^n_{i-1})| + |U_n(t^n_{i-1}) - U_n(t^n_i)| \\
&< \varepsilon_n t^n_i + \varepsilon_n (t^n_i - t^n_{i-1}) = \varepsilon_n t^n_i.
\end{aligned}$$

Also, for $t_i - t < \theta < 0$,

$$W_n(t^n_i)0 + (t + \theta - t_i) f(t^n_i, U_n(t^n_i))$$

$$\begin{aligned}
&= \phi(0) + \int_0^{t^n_i} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau + \int_{t^n_i}^{t^n_i + \theta} f(t^n_i, U_n(t^n_i)) \, d\tau \\
&= \phi(0) + \int_0^{t^n_i + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau \\
&= W_n(t)\theta
\end{aligned}$$

and thus

$$\max_{t^n_i - t < \theta < 0} |W_n(t)\theta - U_n(t)\theta| = |W_n(t^n_i)0 - U_n(t^n_i)0|$$

$$\begin{aligned}
&< |W_n(t^n_i) - U_n(t^n_i)| < \varepsilon_n t^n_i.
\end{aligned}$$

It follows that $|W_n(t) - U_n(t)| \leq \varepsilon_n t^n_i$ for $t \in [t^n_i, t^n_{i+1})$. We also have

$$|U_n(t^n_{i+1}) - \phi| = |U_n(t^n_{i+1}) - W_n(t^n_{i+1}) + W_n(t^n_{i+1}) - \phi|$$

$$\begin{aligned}
&= |\psi + T(t_{i+1})\phi - \phi| \\
\end{aligned}$$

where

$$\psi(\theta) = \begin{cases} 
U_n(t^n_{i+1})\theta - W_n(t^n_{i+1})\theta, 
&{-r < \theta < -t^n_{i+1},} \\
U_n(t^n_{i+1})\theta - W_n(t^n_{i+1})\theta + \int_0^{t^n_{i+1} + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau, 
&{-t^n_{i+1} < \theta < 0},
\end{cases}$$

and so

$$|\psi| < |U_n(t^n_{i+1}) - U_n(t^n_{i+1})| + |U_n(t^n_{i+1}) - W_n(t^n_{i+1})| + t^n_{i+1}M$$

$$\begin{aligned}
&< \varepsilon_n (t^n_{i+1} - t^n_i) + \varepsilon_n (t^n_i) + t^n_{i+1}M \\
&< t^n_{i+1}(M + 1) < b(M + 1).
\end{aligned}$$
By the selection of \( b \) and \( M \) we have
\[
|U_n(t^n_{i+1}) - \phi| = |\psi + T(t^n_{i+1})\phi - \phi| \leq R.
\]
It readily follows that \( U_n \) satisfies (i)-(v) on \([0, t^n_{i+1}]\). To establish that there exists \( N \) such that \( t^n_i = b \), suppose for contradiction that \( t^n_i < b \) for all \( i \) and \( \lim_{i \to \infty} t^n_i = c < b \). For each \( k = 0, 1, 2, \ldots \) define \( W^k_n : [t^n_k, c] \to \mathbb{R} \) by
\[
W^k_n(t) = \begin{cases}
\left[ U_n(t^n_k) \right](t - t^n_k + \theta), & -r < \theta < t^n_k - t,
U_n(t^n_k)0 + \int_{t^n_k}^{t^n_k + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & t^n_k - t < \theta < 0.
\end{cases}
\]
Using an argument similar to that employed above we have for \( t^n_i > t^n_k \) that
\[
|W^k_n(t^n_i) - W^k_n(t^n_k)| \leq e_n(t^n_i - t^n_k) < e_n(c - t^n_k).
\]
Also, for \( j > l \)
\[
W^k_n(t^n_j) - W^k_n(t^n_l) = \begin{cases}
\left[ U_n(t^n_k) \right](t^n_j - t^n_k + \theta) - \left[ U_n(t^n_k) \right](t^n_l - t^n_k + \theta), & -r < \theta < t^n_k - t^n_l,
U_n(t^n_k)0 + \int_{t^n_l}^{t^n_j + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau - \left[ U_n(t^n_k) \right](t^n_j - t^n_l + \theta), & t^n_l - \theta < \theta < t^n_k - t^n_l,
\int_{t^n_l}^{t^n_j + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau - \int_{t^n_l}^{t^n_l + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & -r < \theta < 0,
t^n_l - \theta < \theta < 0,
\end{cases}
\]
from which one can show that \( \{ W^k_n(t^n_j) \}_{i=k}^{\infty} \) is a Cauchy sequence. It follows that \( \lim_{i \to \infty} U_n(t^n_i) \) exists and since \( \Omega \) is closed we have \( \lim_{i \to \infty} U_n(t^n_i) = W \in \Omega \). Since \( \{ U_n(t^n_i) \}_{i=1}^{\infty} \cup \{ W \} \) is compact, by the continuity of \( f \) and the definition of \( T(h) \) there exists \( \delta > 0 \) such that if \( h \in [0, \delta] \) then \( h \) may replace \( \delta^n_i \) in (a). Also, since \( \{ U_n(t^n_i) \}_{i=1}^{\infty} \cup \{ W \} \) is compact it is an equicontinuous family so we may assume if \( h \in [0, \delta] \) then \( h \) may replace \( \delta^n_i \) in (b). However, there exists \( h_0 \in [0, \delta] \) such that \( |F_{h_0}(c, W); \Omega| \leq h_0\delta_n/3 \) and since for sufficiently large \( i \) we have \( \delta^n_i < h_0 \) and \( \delta^n_i \) is the maximum number satisfying (a)-(c), it must be that \( |F_{h_0}(t^n_i, U_n(t^n_i)); \Omega| \geq h_0\delta_n/2 \). But this is impossible since
\[
h_0\delta_n/3 > |F_{h_0}(c, W); \Omega| = \lim_{i \to \infty} |F_{h_0}(t^n_i, U_n(t^n_i)); \Omega| \geq h_0\delta_n/2.
\]
We conclude that there exists \( N \) such that \( t^n_N = b \). This completes the proof of the lemma.

**Proof of the theorem (Continued).** We now show that (2) \( \rightarrow \) (1). We first establish that there exists a continuous function \( U : [0, b] \to \Omega \) such that \( \lim_{n \to \infty} U_n = U \) uniformly on \([0, b]\).
Let $W_n$ be defined by (iv) in the statement of the lemma and write, for $0 < s < t < b$,

$$[W_n(t) - W_n(s)]\theta = \begin{cases} 
\phi(t + \theta) - \phi(s + \theta), & -r < \theta < -t, \\
\phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau - \phi(s + \phi), & -t < \theta < -s, \\
\int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau - \int_0^{s+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau, & -s < \theta < 0.
\end{cases}$$

With this representation one can show that $(W_n)_{n=1}^{\infty}$ forms an equicontinuous family. To apply Ascoli's theorem to $(W_n)_{n=1}^{\infty}$ we must show for each $t \in [0, b]$ that $(W_n(t))_{n=1}^{\infty} \subset C$ is precompact. However, similar to the above one can show for each $t \in [0, b]$ that $(W_n(t))_{n=1}^{\infty}$ is an equicontinuous family of $C$ and since $|W_n(t)| < |\phi| + bM$ for all $1 < n < \infty$ we have by Ascoli's theorem that $(W_n(t))_{n=1}^{\infty}$ is precompact and consequently $(W_n)_{n=1}^{\infty}$ is precompact. Relabelling, if necessary, we have that there exists a continuous function $U: [0, b] \to C$ such that $U = \lim_{n \to \infty} W_n$ uniformly on $[0, b]$. For each $t \in [t^n, t^n_{+1})$, writing

$$[U_n(t) - U_n(t^n)]\theta = \begin{cases} 
[U_n(t^n)](t - t^n + \theta) - [U_n(t^n)]\theta, & -r < \theta < t^n - t, \\
U_n(t^n)0 + (t - t^n + \theta)f(t^n, U_n(t^n)) - U_n(t^n)\theta, & t^n - t < \theta < 0,
\end{cases}$$

and using (iii) of the lemma, we obtain $|U_n(t) - U_n(t^n)| < \varepsilon_n(1 + M)$. Also, for $t \in [t^n, t^n_{+1})$ by (iv) we see $|W_n(t) - U_n(t)| < \varepsilon_n t^n < \varepsilon_n b$ and thus $U(t) = \lim_{n \to \infty} U_n(t) = \lim_{n \to \infty} U_n(\gamma_n(t)) \in \Omega$ uniformly on $[0, b]$. The continuity of $f$ and compactness of $(t, U(t)): 0 < t < b$ yields

$$\lim_{n \to \infty} f(\gamma_n(t), U_n(\gamma_n(t))) = f(t, U(t))$$

uniformly on $[0, b]$. Let $t \in [0, b]$. Then, if $\theta \in [-r, t]$,

$$U(t)\theta = \lim_{n \to \infty} W_n(t)\theta = \phi(t + \theta)$$

and, if $\theta \in [-t, 0]$,

$$U(t)\theta = \lim_{n \to \infty} W_n(t)\theta = \lim_{n \to \infty} \phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) \, d\tau$$

$$= \phi(0) + \int_0^{t+\theta} f(\tau, U(\tau)) \, d\tau.$$
we have that \( x \) is a solution to (\(*\)) with \( x_t = U(t) \in \Omega \) for each \( t \in [0, b] \). This completes the proof of the theorem.

**Remark 1.** Let \( \Lambda \subset \mathbb{R}^n \) and \( \Omega(\Lambda) \subset \mathcal{C} \) be defined by \( \Omega(\Lambda) = \{ \phi \in \mathcal{C}: \phi(\theta) \in \Lambda \text{ for } \theta \in [-r, 0] \} \). Note that for \( t > 0 \), \( x_t \in \Omega(\Lambda) \) if and only if \( x(s) \in \Lambda \) for \( s \geq r \). Also, in the case that \( \Lambda \) is convex (2) of the theorem is equivalent to

\[
(2)' \text{ for each } (t, \phi) \in [0, \infty) \times \Omega(\Lambda), \liminf_{h \to 0^+} h^{-1} \phi(0) + h f(t, \phi) ; \Lambda \cap A / h = 0.
\]

That (2) \( \rightarrow \) (2)' is evident. Suppose (2)' holds, let \( \varepsilon > 0 \), and let \( (t, \phi) \in [0, \infty) \times \Omega(\Lambda) \). By (2)' there exist \( h > 0 \) and \( x_h \in \Lambda \) such that \( |\phi(0) + h f(t, \phi) - x_h| < \varepsilon h / 2 \).

Define \( \psi_h \in \Omega(\Lambda) \) by

\[
\psi_h(\theta) = \begin{cases} 
\phi(\theta + h), & -r < \theta < -h, \\
\frac{1}{h} (\theta + h)x_h - \theta \phi(0), & -h < \theta < 0.
\end{cases}
\]

Then

\[
\frac{1}{h} |F_h(t, \phi); \Omega(\Lambda)| \leq \frac{1}{h} |F_h(t, \phi) - \psi_h |
\]

\[
= \max_{-h < \theta < 0} \frac{1}{h} |\phi(0) + (h + \theta) f(t, \phi) - \frac{\theta + h}{h} x_h - \phi(0)|
\]

\[
\leq \max_{-h < \theta < 0} \frac{1}{h} (1 + |\theta|) |\phi(0) + h f(t, \phi) - x_h| < \varepsilon
\]

and it follows that (2)' \( \rightarrow \) (2). Consequently, the techniques in the theorem yield results in Seifert [6]. Actually, in [6], \( \mathcal{C} \) is replaced by \( \mathcal{C} \otimes \mathcal{S} = \{ \phi: \mathbb{R} \to \mathbb{R}^n: \phi \text{ is continuous and bounded} \} \). However, the necessary modifications to the theorem are easily performed.

**Remark 2.** Suppose we replace \( \mathbb{R}^n \) with a general Banach space \( \mathcal{X} \). This generalization requires no change in the lemma. The difficulty arises in the proof of the theorem in which we show (2) \( \rightarrow \) (1) by obtaining the convergence of a subsequence of \( \{ W_n \}_{n=1}^{\infty} \); in particular, we are not assured for each \( (t, \theta) \in [0, b] \times [-t, 0] \) that \( \{ W_n(t, \theta) \}_{n=1}^{\infty} \) is precompact. This can obviously be overcome if we assume that \( f \) maps bounded subsets of \( [0, b] \times \Omega \) into precompact subsets of \( \mathcal{X} \). We may also obtain that \( \{ W_n(t, \theta) \}_{n=1}^{\infty} \) is precompact by supposing that

\[
(3) f \text{ is uniformly continuous on } [0, b] \times \Omega \text{ and for each bounded set } P \subset \Omega \text{ and for each bounded set } P \subset \Omega \text{ and for each bounded set } P \subset \Omega
\]

\[
\lim_{h \to 0} \left( \alpha \left( \{ \psi(0) + h f(t, \psi) \}_{\psi \in P} \right) - \alpha \left( \{ \psi(0) \}_{\psi \in P} \right) / h \right) \leq g(t, \alpha \left( \{ \psi(0) \}_{\psi \in P} \right)
\]

whenever \( \alpha \left( \{ \psi(0) \}_{\psi \in P} \right) \leq \alpha \left( \{ \psi(0) \}_{\psi \in P} \right) \) for \( \theta \in [0, b] \times [-t, 0] \). Here \( g: \mathbb{R} \to \mathbb{R} \) is such that one obtains an estimate on solutions to \( x' = (t) \leq g(t, x(t)) \); for a bounded set \( A \subset \mathcal{X} \), \( \alpha[A] \) denotes the measure of noncompactness of \( A \); and \( \alpha[\{ \psi(0) \}_{\psi \in P}] = \alpha[\{ \psi(0): \psi \in P \}] \). Assumption (3) is made by Leela and Moauro [2] and is motivated by a similar condition used for ordinary differential equations by Li [3] and Martin [5]. The resulting
proof that \( \{ W_n(t) \theta \}_{n=1}^{\infty} \) is precompact is straightforward given the techniques used in those papers. In [2], \( \Omega = \Omega(\Lambda) \) for some closed \( \Lambda \subset X' \) and the invariance assumption made on \( f \) is

\[
(4) \liminf_{h \to 0^+} \frac{|\phi(0) + hf(t, \phi); \Lambda|}{h} = 0
\]

for all \((t, \phi) \in [0, \infty) \times C\) with \( \phi(0) \in \Lambda \). The following simple example typifies a situation in which our results apply but those of [2] do not and so our results actually improve those of [2].

Let \( \Lambda = [-1, 0] \) and \( f: C \to \mathbb{R} \) be defined by \( f(\phi) = \phi(-r) \phi(0) \). It is easy to verify that for \( \phi \in \Omega(\Lambda) \), \( F_h(\phi) \in \Omega(\Lambda) \) for sufficiently small \( h \) and so (2) holds. However, if \( \phi(\theta) = -(2/r)\theta - 1 \) then \( \phi(0) = -1 \in \Lambda \) (note that \( \phi \not\in \Omega(\Lambda) \)) but \( f(\phi) = -1 \) and thus

\[
\liminf_{h \to 0^+} \frac{|\phi(0) + hf(\phi); \Lambda|}{h} = \liminf_{h \to 0^+} \frac{|-1 - h; \Lambda|}{h} = 1.
\]

REFERENCES


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