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OPERATORS COMMUTING WITH POSITIVE OPERATORS

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ABSTRACT. Necessary and sufficient conditions are obtained for an operator to commute with a positive operator.

Throughout the paper, by an operator we mean a bounded linear transformation acting in a Hilbert space H. The algebra of all operators in H is denoted by $B(H)$.

Arveson's theorem [1] about transitive algebras states that if $\mathfrak{A}$ is a strongly closed transitive algebra of operators and if $\mathfrak{A}$ contains a maximal abelian selfadjoint algebra (with respect to $B(H)$), then $\mathfrak{A} = B(H)$. (A transitive algebra is one whose only invariant subspaces are $\{0\}$ and H.) Foiaş [2] gives a different proof of Arveson's theorem mainly based on the following facts:

(F1) If $\mathfrak{A}$ is a strongly closed proper subalgebra of $B(H)$, then $\mathfrak{A}$ leaves the range of a nonzero, noninvertible positive operator K invariant. In particular, if $\mathfrak{A}$ contains a maximal abelian selfadjoint algebra $\mathfrak{R}$, then K can be chosen such that $K \in \mathfrak{R}$ and $\mathfrak{A}K \subset K\mathfrak{A}$.

(F2) If $\mathfrak{A}$ is a uniformly closed algebra and $\mathfrak{A}K \subset K\mathfrak{A}$ for some noninvertible positive operator $K \neq 0$, then $\mathfrak{A}$ is not transitive.

In the proof of (F2) it is shown that if E is the resolution of the identity for K and $T \in \mathfrak{A}$, then $TE([t, \infty))H \subset E([t/a, \infty))H$ for $0 < t < \|K\|$, where a is a fixed number not less than 1. (For a similar result about decomposable operators see [3].) In the present paper, given a positive operator K, we study conditions on an arbitrary T (not necessarily in an algebra) to satisfy the above condition, and also prove a kind of converse to our result. In fact, Theorem 1 shows that if T is an operator and $a \geqslant 1$, and if

$$\liminf_{n>0} (\|K^{-n}TK^n\|/a^n) < \infty, \tag{1}$$

then

$$TE([t, \infty))H \subset E([t/a, \infty))H, \qquad 0 < t < \|K\|. \tag{2}$$

Note that we assume $K^{-n}TK^n$ can be extended boundedly to all of H. In (1) and (2), K is an injective positive operator and E is its resolution of the identity. Conversely, Theorems 2 and 3 show that if T satisfies (2), then (1) holds but for a replaced by a^2. As corollaries, we obtain necessary and

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sufficient conditions for an operator to commute with an injective positive operator.

THEOREM 1. *Let K be an injective positive operator and let $a > 1$. Then the following assertions are true.*

(a) *For every operator T, condition* (1) *implies condition* (2).

(b) *If condition* (2) *holds for all operators T in some algebra $\mathfrak{A}$, then for each $t \in (0, \|K\|)$ the closure of $\mathfrak{A}E([t, \infty))H$ is an invariant subspace of $\mathfrak{A}$, which is nontrivial if $0 \in \sigma(K)$ and $I \in \mathfrak{A}$.*

(c) (Foiaş) *If $\mathfrak{A}K \subset K\mathfrak{A}$ for some uniformly closed algebra $\mathfrak{A}$, then for each $t \in (0, \|K\|)$ the closure of $\mathfrak{A}E([t, \infty))H$ is an invariant subspace of $\mathfrak{A}$ and K, which is nontrivial if $0 \in \sigma(K)$ and $I \in \mathfrak{A}$.*

PROOF. Assume without loss of generality that $\|K\| = 1$ and $0 < t \leq 1$.

(a) Let $b < 1/a$ and let $K = K_1 \oplus K_2 \oplus K_3$ with respect to some orthogonal direct sum $H = H_1 \oplus H_2 \oplus H_3$ such that $\sigma(K_1) \subset [t, 1]$, $\sigma(K_2) \subset [tb, t]$ and $\sigma(K_3) \subset [0, tb]$. Note that some of these spaces may be trivial. If $H_3 \neq \{0\}$, let $T_{31} = P_3T|H_1$, where P_3: $H \to H_3$ is the projection onto H_3. It follows that

$$\begin{aligned}\|T_{31}\| &= \|K_3^nK_3^{-n}T_{31}K_1^nK_1^{-n}\| \\ &\leq \|K_3^n\|\cdot\|K_1^{-n}\|\cdot\|K_3^{-n}T_{31}K_1^n\| \leq \|K^{-n}TK^n\|b^n.\end{aligned}$$

Thus $\|T_{31}\| \leq \liminf_{n>0}\|K^{-n}TK^n\|b^n = 0$, and hence $TH_1 \subset H_1 \oplus H_2 \subset E([tb, 1])H$. Therefore $TE([t, 1])H \subset E([tb, 1])H$ for all $b < 1/a$ and thus (2) follows.

(b) assume without loss of generality that $\mathfrak{A}$ contains the identity. Then for $t \in (0, 1)$ the closure of $\mathfrak{A}E([t, 1])H$ is a nonzero invariant subspace of $\mathfrak{A}$ (included in $E([t/a, 1])H$).

(c) Here, again, assume without loss of generality that $\mathfrak{A}$ contains the identity. Since $\mathfrak{A}$ is uniformly closed and $K^{-1}TK \in \mathfrak{A}$ for all $T \in \mathfrak{A}$, it follows from the closed graph theorem that the map $W(T) = K^{-1}TK$ is a bounded operator in $\mathfrak{A}$ and hence

$$\limsup_{n>0}\|K^{-n}TK^n\|/\|W\|^n < \infty \quad \text{for all } T \in \mathfrak{A}.$$

For a fixed $t \in (0, 1)$ let $M = \mathfrak{A}E([t, 1])H$. In view of (b), the closure of M is an invariant subspace of $\mathfrak{A}$. Let $x \in E([t, 1])H$ and let $T \in \mathfrak{A}$. Let $K_1 = K|E([t, 1])H$ and $K_2 = K|E([ta, t))H$. Since $K_1^{-1}x \in E([t, 1])H$, it follows that $(K_1 \oplus K_2)^{-1}Tx = (K_1 \oplus K_2)^{-1}TKK_1^{-1}x = K^{-1}TKK_1^{-1}x \in M$ and thus $(K_1 \oplus K_2)^{-1}M \subset M$. Hence the closure of M is an invariant subspace of $(K_1 \oplus K_2)^{-1}$ and thus of $K_1 \oplus K_2$, because $K_1 \oplus K_2$ is Hermitian.

COROLLARY 1. *Let K be a nonscalar, injective positive operator and assume* $\liminf_{n>0}\|K^{-n}TK^n\| < \infty$ *for some operator T. Then T has a nontrivial invariant subspace.*

THEOREM 2. *Let K be an injective positive operator with the resolution of the identity E. Assume* (2) *holds for some operator T and some $a > 1$. Then*

$$\|K^{-n}TK^n\| \leqslant (a^{3n}/(a^n - 1))\|T\| \qquad (n = 1, 2, 3, \dots).$$

PROOF. Assume without loss of generality that $\|K\| = 1$. Let $b = 1/a$. Let $H_1 = E([b, 1])H$ and $H_i = E([b^i, b^{i-1}))H$ for $i = 2, 3, \dots$. Note that some H_i may be trivial. Let $J = \{i\colon H_i \neq \{0\}\}$. For $i, j \in J$, let $K_i = K|H_i$ and $T_{ij} = P_i T|H_j$, where $P_j\colon H \to H_j$ is the projection onto H_j. By the hypotheses, $T_{ij} = 0$ for $i \geqslant j + 2$. For $i \leqslant j + 1$ and $i, j \in J$ we have

$$K_i^{-n}T_{ij}K_j^n = P_jK^{-n}TK^n|H_i,$$

$$\|K_i^{-n}T_{ij}K_j^n\| \leqslant b^{-ni}b^{nj-n}\|T\| = b^{n(j-i-1)}\|T\|,$$

for $n = 1, 2, \dots$. Let $C = ((c_{ij}))$ be the matrix in which

$$c_{ij} = 0 \quad \text{if } i \geqslant j + 2,$$

$$c_{ij} = \|T\|b^{n(j-i-1)} \quad \text{if } i < j + 2.$$

Obviously

$$\|T\|^{-1}C = b^{-2n}S + b^{-n}I + \sum_{0<k} b^{nk}(S^*)^{k+1},$$

where S is a unilateral shift. Hence $\|C\| \leqslant \|T\|b^{-2n}/(1 - b^n)$. Since $K^{-n}TK^n = ((K_i^{-n}T_{ij}K_j^n))_{i,j\in J}$ is majorized by the compression $((c_{ij}))_{i,j\in J}$ of $((c_{ij}))$, it follows from [3, Lemma 1] that

$$\|K^{-n}TK^n\| \leqslant \frac{b^{-2n}}{1 - b^n}\|T\| = \frac{a^{3n}}{a^n - 1}\|T\|$$

for $n = 1, 2, \dots$.

The interesting case of $a = 1$ is treated in the following theorem and corollaries.

THEOREM 3. *Let K be an injective positive operator with the resolution of the identity E and let T be an arbitrary operator. Assume $TE([t, \infty)) = E([t, \infty))TE([t, \infty))$ for all $t > 0$. Then $\|K^{-n}TK^n\| \leqslant 4\|T\|$ for $n = 1, 2, \dots$.*

PROOF. Assume without loss of generality that $\|K\| = 1$. Let $b = 2^{-1/n}$ and let H_i, K_i, and T_{ij} be as in the proof of Theorem 2. Here $T_{ij} = 0$ for $i \geqslant j + 1$. Following the proof of Theorem 2, we obtain

$$\|K^{-n}TK^n\| \leqslant (b^{-n}/(1 - b^n))\|T\| = 4\|T\|, \qquad n = 1, 2, \dots.$$

COROLLARY 2. *Let K be an injective positive operator with the resolution of the identity E, and let T be an arbitrary operator. Then $TE([t, \infty)) = E([t, \infty))TE([t, \infty))$ for all $t > 0$ if and only if $\|K^{-n}TK^n\|$ is uniformly bounded for $n = 1, 2, \dots$.*

COROLLARY 3. *Let K be an injective positive operator and let T be an arbitrary operator. Then $TK = KT$ if and only if $\|K^{-n}TK^n\|$ is uniformly bounded for* $n = \pm 1, \pm 2, \pm 3, \ldots$.

The last corollary has the following generalization.

COROLLARY 4. *Let K and L be two injective positive operators and let T be an arbitrary operator. Then $KT = TL$ if and only if $\|K^{-n}TL^n\|$ is uniformly bounded for* $n = \pm 1, \pm 2, \pm 3, \ldots$.

PROOF. Consider the operators

$$\begin{pmatrix} K & 0 \\ 0 & L \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix},$$

and apply Corollary 3.

ADDED IN PROOF (SEPTEMBER 1978). A special case of Theorem 3 in which K is boundedly invertible is proved by James A. Deddens in a paper entitled *Another description of nest algebras* to appear in Proc. Long Beach Conf., June 1977, Springer, New York. Our proof given here is much shorter. Using Deddens' result, J. P. Williams obtains our Corollary 3 for invertible K.

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