AN EXTENSION OF THE HARDY-LITTLEWOOD INEQUALITY

M. K. KWONG AND A. ZETTL

Abstract. The Hardy-Littlewood inequality is extended from $L^2$ to $L^2_w$ where $w$ is any positive nondecreasing function.

In this note we establish the inequality

$$\left( \int_J |y'|^2 w \right)^2 < 4 \int_J |y|^2 w \int_J |y''|^2 w$$

for any complex valued function $y$ satisfying

$$y \in L^2_w(J), \ y' \text{ locally absolutely continuous, } y'' \in L^2_w(J).$$

Here $J = (0, \infty)$ or $J = (-\infty, \infty)$ and $w$ is any positive nondecreasing function on $J$. The case $w(t) \equiv 1$ is the inequality of the title; in this case the constant 4 is best possible when $J = (0, \infty)$ but can be replaced by 1 when $J = (-\infty, \infty)$ [3].

Proof. Suppose $y$ satisfies (2). First we show that

$$y^2(t)w(t) \to 0 \quad \text{as } t \to \infty.$$  

Since $w(t)$ is nondecreasing we have

$$w(t) \int_t^\infty |y'(i)|^2 < \int_t^\infty |y(i)|^2 w, \quad i = 0, 2.$$  

Hence $y$ and $y''$ are in $L^2(t, \infty)$ and by a well-known result [2] relating the supremum norm of $y'$ to the $L^2$ norms of $y$ and $y''$ we have

$$|y(t)|^2 < \|y\|_{2,(t, \infty)}^2 < K\left( \int_t^\infty |y|^2 \right)^{1/4} \left( \int_t^\infty |y''|^2 \right)^{3/4} \leq Kw^{-1}(t)\left( \int_t^\infty |y|^2 w \right)^{1/4} \left( \int_t^\infty |y''|^2 w \right)^{3/4}.$$  

Therefore $w(t)y^2(t) \to 0$ as $t \to \infty$.

Now we show that the differentiation operator $A$ defined by $Ay = y'$ is dissipative on $L_w(0, \infty)$. We have, for $y = u + iv$,

$$\text{Re}(Ay, y) = \int_0^\infty (u'u + v'v)w$$

Received by the editors October 23, 1978.


Keywords and phrases. Landau, Hardy-Littlewood inequality, norm inequality for derivatives.

The research of this author was partially supported by NSF grant MCS76-06623 A01.

© 1979 American Mathematical Society

0002-9939/79/0000-0474/50.00
and
\[ 2 \int_0^t u' w = u^2(t)w(t) - u^2(0)w(0) - \int_0^t u^2 \, dw. \] (5)

From (3) and (5) we conclude that Re(Ay, y) < 0 for all y satisfying (2), i.e., y in the domain of A² on L²(0, ∞) and hence A is dissipative. Since every dissipative operator on Hilbert space has an m-dissipative extension [1] inequality (1) follows from Kato's inequality for m-dissipative operators [4]. The proof for J = (−∞, ∞) is entirely similar.

Inequality (1) does not hold for arbitrary weight functions w as can be seen from the simple example: y(t) = t, w(t) = e^{-t}, 0 < t < ∞.

Kato in [4] also characterizes the cases of equality. From this characterization follows that there is equality in (1) if and only if
\[ \int_J f_{b,c} f'_{b,c} w = 0 \] (6)
for some constants b, c, where
\[ f_{b,c}(t) = \exp(-bt/2)\sin(\sqrt{3} bt/2 - c). \]

When w = 1 and J = (0, ∞), all extremals of (1) are given by af_{b,c} with \( c = \pi/3 \) and a, b constants with b > 0. It can be seen from (6) that there are nonconstant weight functions w for which (1) has extremals yielding the best constant 4 in both cases J = (0, ∞) and J = (−∞, ∞). For instance \( w(t) = 1 \) in [0, 4π/√3] and \( w(t) = 2 \) for \( t > 4\pi/\sqrt{3} \) has extremal \( f(t) = \exp(-t/2)\sin(\sqrt{3} t/2 - \pi/3) \)
giving the constant 4 in (1). Of course when (6) does not hold the best constant in (1) may be less than 4.

REFERENCES


DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115