CONSTRUCTION OF A RIGID ARONSZAJN TREE

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Abstract. We construct in ZFC an Aronszajn tree with no automorphism.

Using the nonisomorphic Aronszajn trees constructed by Gaifman and Specker [1] we build in ZFC a normal rigid Aronszajn tree thus answering a question of T. J. Jech [2].

For a tree $T$ and $a \in T$, $Ta$ denotes the part of $T$ above $a$, i.e., $Ta = \{ x \in T | x > a \}$. $T_a$, $a < \omega_1$, is the $a$th level of $T$. We say that $a, b \in T$ meet at the level $a$ if there is an $e \in T_a$, $e < a$, $e < b$, but no such $e$ exists in $T_{a+1}$.

By [1] we have for every uncountable $X \subseteq \omega_1$ a normal Aronszajn tree $T(X)$ such that:
(1) for every $a \in T(X)$ there is an uncountable subset of $T(X)_a$ every two elements of which meet at level in $X$, and
(2) there does not exist an uncountable subset of $T(X)$ every two elements of which meet at level in $\omega_1 - X$.

(Briefly, if we start from an Aronszajn tree $T \subseteq \omega_\omega$ then $T(X) \subseteq \omega_\omega$ is the set of all functions $f$ such that the collapse of $f|X$ is in $T$ and $f(\omega_1 - X)$ is different from zero only finitely many times.)

We will construct a normal Aronszajn tree $R$ such that there is no order preserving function $f$: $R \rightarrow R$ other than the identity. Let $X$, $X_{a,n}$, $a < \omega_1$, $n < \omega$, be pairwise disjoint uncountable subsets of $\omega_1$, construct the trees $T(X_{a,n})$ such that their domain will be pairwise disjoint. We construct a sequence $R^n$, $n < \omega$, of Aronszajn trees such that $R^n \subseteq R^{n+1}$ by induction on $n$. A point in $R^{n+1} - R^n$ is said to be of rank $n + 1$.

$R^0$ is $T(X)$. Suppose $R^n$ is defined, let $\{ a^a_\gamma | \gamma < \omega_1 \}$ be its elements, for every $a^a_\gamma$, $a^a_\gamma \in R^n_a$ for some $\alpha < \omega_1$, take any member $b^n_\gamma$ of $T(X_{a,n})_{a+1}$ and plant $T(X_{a,n})_{b^n_\gamma}$ above $a^a_\gamma$. The result is $R^{n+1}$. Again, $R^{n+1}$ is obtained by partial ordering $R^n \cup \gamma < \omega_1$, $T(X_{\gamma,n})_{b^n_\gamma}$ as follows: for elements of $R^n$ or $T(X_{\gamma,n})$ the order is as in $R^n$ or $T(X_{\gamma,n})$, respectively, $b^n_\gamma$ is above $a^a_\gamma$ and the partial order is transitive. For $x \in T(X_{\gamma,n})$ we define $r(x) = b^n_\gamma$ and $r^*(x) = a^a_\gamma$, for $x \in R^n$, $r(x)$ is the root of $R^n$.

$R = \bigcup n < \omega R^n$ is a normal Aronszajn tree. Note that for $m < n$ an element of $R^n$ cannot be below an element of rank $m$. We prove that $R$ is

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1ADDED IN PROOF. This result was already known to J. Baumgartner. See Notices Amer. Math. Soc. 22 (1975), p. A-219.

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rigid. Let \( f : R \rightarrow R \) be order preserving and for some distinct \( x, y \in R_\alpha, f(x) = y \).

For some \( n < \omega, x, y \in R^n, x = a^\gamma_n \) for some \( \gamma < \omega_1 \) and \( T(X_{\gamma,n})b^\gamma_n \) was grafted above \( a^\gamma_n \), it follows from (1) that we can find \( \aleph_1 \) elements in \( R_{f(x)} \) every two of which meet at level in \( X_{\gamma,n} \). Let \( k < \omega \) be the least integer such that we can find an uncountable \( A \subseteq R^k_y \) every two elements of which meet at level in \( X_{\gamma,n} \). Look at \( B = \{r^*(x) | x \in A \} \) for such an \( A \). If it is uncountable then we have \( \aleph_1 \) elements in \( R^k_y \) contradicting the definition of \( k \). Hence we can find \( b \) such that \( b = r(x) \) for uncountably many \( x \) in \( A \). \( b \) is compatible with \( y \), hence distinct from \( x \) so that \( b = b^\eta_n \) for some \( (\eta, l) \) distinct from \( (\gamma, n) \) it follows that we have found \( \aleph_1 \) elements in \( T(X_{n,l}) \) every two of which meet at level outside \( X_{n,l} \), and this is a contradiction.

REFERENCES


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