SHORTER NOTES

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NONTRIVIALLY PSUEDOCOMPLEMENTED LATTICES
ARE COMPLEMENTED

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ABSTRACT. Nontrivially pseudocomplemented lattices are complemented.

A lattice \( L \) with 0 and 1 is called pseudocomplemented if for each \( a \in L \) there is a largest \( a^* \) with \( a \land a^* = 0 \). Such an \( a^* \) is called the pseudocomplement of \( a \). A pseudocomplemented lattice is called nontrivially pseudocomplemented if each nonunit element has a nonzero pseudocomplement.

THEOREM. Nontrivially pseudocomplemented lattices are complemented.

Proof. Let \( L \) be a nontrivially pseudocomplemented lattice. First observe that the intersection of ultrafilters (maximal dual ideals of some authors) of \( L \) is equal to \{1\}. Indeed, for any nonunit element \( a \) of \( L \), a Zorn's Lemma argument shows the existence of a filter (dual ideal) maximal subject to the conditions (i) containing the (proper) principal ideal \( \{ x \in L: x > a^* \} \) and (ii) not containing \( a \). It is easy to see that this filter is an ultrafilter of \( L \) since any larger filter either coincides with it or is equal to \( L \) according to whether \( a \) belongs to or does not belong to the larger filter. Clearly this ultrafilter does not contain \( a \).

Now let \( b \in L \) be arbitrary. We shall show that \( b \lor b^* = 1 \) or, equivalently, \( b \lor b^* \) belongs to all ultrafilters of \( L \) and thus \( b^* \) is a complement of \( b \). Let \( U \) be an arbitrary ultrafilter of \( L \). If \( b \in U \) then so is \( b \lor b^* \), so we may assume that \( b \notin U \). By the maximality of \( U \), the filter generated by \( b \) and \( U \) contains 0, so there is an \( x \) in \( U \) with \( b \land x = 0 \). But \( b^* \) is the largest element whose meet with \( b \) equals zero, so \( b^* \) dominates the element \( x \) of (the filter) \( U \), showing that it is itself in \( U \). Clearly again \( b \lor b^* \) is in \( U \) and the rest follows.

Remarks. (1) The nontriviality assumption is essential. For example chains are (trivially) pseudocomplemented but not necessarily complemented.

(2) A complete lattice is called \( \land \)-distributive if, for any \( b \in L \) and any subset \( A \) of \( L \), \( b \land \lor A = \lor \{ b \land a: a \in A \} \). A lattice is called weakly...
disjunctive if for each nonunit element $a$ of $L$ there is a nonzero element $b$
with $a \land b = 0$. Clearly complete, $\land$-distributive weakly disjunctive lattices
are nontrivially pseudocomplemented but the converse is false as seen by
considering the five element nonmodular pentagon. In [2] MacNab shows
that complete, $\land$-distributive weakly disjunctive lattices are Boolean alge-
bras. The proof in [2] uses distributivity to show that the lattice is com-
plemented. By the above theorem this follows easily without any use of distribu-
tivity and gives MacNab’s result as a corollary.

(3) In [1] we strengthen a result of Tarski by characterizing complete
atomic Boolean algebras as semisimple complete and completely distributive
lattices. As pointed out by MacNab in [2], this follows from the theorem in [2]
(and so from the theorem here) using Tarski’s result. Notice however that the
proof in [1] strengthens Tarski’s result without using it.

REFERENCES

1. M. S. Lambrou, *Semisimple complete and completely distributive lattices are Boolean algebras*,

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